We study strategyproof (SP) mechanisms for the location of a facility on a discrete graph. We give a full characterization of SP mechanisms on lines and on sufficiently large cycles. Interestingly, the characterization deviates from the one given by Schummer and Vohra [2004] for the continuous case. In particular, it is shown that an SP mechanism on a cycle is close to dictatorial, but all agents can affect the outcome, in contrast to the continuous case. Our characterization is also used to derive a lower bound on the approximation ratio with respect to the social cost that can be achieved by an SP mechanism on certain graphs. Finally, we show how the representation of such graphs as subsets of the binary cube reveals common properties of SP mechanisms and enables one to extend the lower bound to related domains.

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guaranteed by an SP mechanism. This agenda, often termed approximate mechanism design without money, was recently advocated by Procaccia and Tennenholtz [2009].

The facility location problem is a very natural setting, in which agents whose ideal locations are located in some metric space report their ideal locations (which is their private information), and a facility location is determined based on these reports, with some objective function in mind. For example, one might wish to locate the facility at a point that will minimize the average distance to an agent. This family of problems arises in many real-life scenarios, such as locating a library or a bus station in a street. Actually, this problem arises also in more virtual scenarios, such as deciding on the salary of a manager during a board meeting of a firm. For simple objective function, the problem is trivial if truthfulness can be assumed. However, a naive mechanism may fail if agents act strategically. For example, when considering where to locate a bus station in a street, it is fairly easy to see that a mechanism that picks the average point of all reported locations can be easily manipulated. In particular, an agent can bias the chosen facility location toward her ideal location by misreporting it.

1.1. Previous work

While the most studied setting for Mechanism design without money (MDw/oM) is facility location, SP mechanisms without money have been proposed and analyzed in a wide variety of domains such as matching [Schummer and Vohra 2007; Ashlagi et al. 2010; Dughmi and Ghosh 2010], resource allocation [Guo and Conitzer 2010; Guo et al. 2009; Othman et al. 2010], machine learning [Dekel et al. 2010; Meir et al. 2011], judgment aggregation [Dietrich and List 2007b; Nehring and Puppe 2007], and even auctions [Harrenstein et al. 2009].

The deterministic facility location problem on a continuous graph was studied by Schummer and Vohra [2004]. They characterized deterministic SP mechanisms on a line, and extended the characterization to trees. They further showed that on a circular graph, every SP mechanism (that is also onto) must be dictatorial. That is, the location of the facility always coincides with the location of the dictator.

This result was later leveraged by Alon et al. [2010] to derive a lower approximation bound of $n$ (the number of agents) for the social welfare of any SP facility location problem on a continuous cycle. The authors then demonstrated how a constant approximation can be guaranteed with a randomized mechanism.

There have also been many other studies of the facility location problem and its variations, for example where more than one facility should be placed, or where each agent controls several locations [Lu et al. 2009, 2010]. While most of the deterministic and randomized algorithms are not tailored in particular for continuous or discrete graphs (some in fact work on any metric space), all lower bounds that we are aware of rely on the continuity of the domain quite strongly. It is therefore important to clarify the necessity of the continuity assumption.

Meir et al. [2011, 2012] studied characterizations and approximation bounds for the strategyproof classification problem. While this seems quite unrelated to the problem at hand, a variation of their model (which they call the “realizable” setting) is in fact equivalent to facility location on subgraphs of the binary cube. Meir et al. showed that any deterministic and onto SP mechanism (on some specific domain) must be dictatorial, and proved a similar result for randomized mechanisms. Nevertheless, this strong negative result has to be shown to still hold in the realizable setting, and to the

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1The original, non-realizable setting of Meir et al. can be interpreted as a generalization of the facility location problem, where agents may be placed in locations that are forbidden for the facility. See more details on this mapping in Section 6.
best of our knowledge, no non-trivial lower approximation bound for discrete facility location problem has been derived.

Our model is closely related to the literature on voting with single-peaked preferences. Strategyproof mechanisms in the general single-peaked model on the line [Black 1986] were characterized by Moulin [1980]. Single-peaked preferences on the binary cube have been considered by Barberà et al. [1991] as well as others. Note that the general single-peaked setting allows for richer preference structures, and thus strategyproofness is a more restrictive requirement in such models. Yet, in our model an agent's location implies not only her peak, but her entire preference structure. This special case of single-peakedness is by now standard in the facility location literature, rationalized by the assumption that agents' dissatisfaction is linear in the distance from the selected location. This assumption often holds in various domains whether the distance is geographical, temporal (i.e. time to wait), or virtual (e.g. number of issues with dissatisfactory outcome).

1.2. Our contribution
In this paper, we follow a setting studied by Schummer and Vohra [2004] and later by Alon et al. [2010] for facility location on graphs. We replace the continuous graph in the original model with a discrete unweighted graph, where the agents and the facility are restricted to vertices only. We feel that in practical problems, such as choosing some building in a street to host a facility, a continuous model is inappropriate, and discrete mechanisms should be better understood.

We give an exact characterization of deterministic SP (and onto) mechanisms on certain families of discrete graphs, focusing on a discrete line and discrete circular graphs (cycles). For both families, we give an embedding of the graph as a subset of the binary cube, which interestingly allows us to express sufficient and necessary properties of SP mechanisms using natural notions.

For large cycles, our characterization implies that every onto SP mechanism is nearly-dictatorial. While this result resembles that of Schummer and Vohra for continuous cycles, we show that the size of the cycle matters: for small cycles there are anonymous SP mechanisms that are very far from dictatorial. Further, even for large cycles there exist SP mechanisms in which all the agents have some level of influence. As a corollary, we get the first lower approximation bound on discrete facility location and show it to be linear in the number of agents. This result also entails a similar lower bound for realizable SP classification problems, thereby showing that the negative result of Meir et al. [2012] also holds in a particular case of interest.

Longer proofs are deferred to the appendix, to allow continuous reading.

2. PRELIMINARIES
Consider a graph \( G = (V, E) \) with a set \( V \) of vertices and a set \( E \) of edges, where edges have no weights nor direction. The vertices \( v \in V \) will be also referred to as the locations, and the two terms will be used interchangeably. The distance between two vertices \( v, v' \in V \), denoted \( d(v, v') \), is the length of the minimum-length path between \( v \) and \( v' \), where the length of a path is defined as the number of edges in the path.\(^2\)

Note that \( d \) is a distance metric. We extend the notion of distance between vertices to distance between sets of vertices, where the distance between two sets of vertices \( A, A' \subseteq V \), denoted \( d(A, A') \) is defined as \( d(A, A') = \min_{v \in A, v' \in A'} d(v, v') \). In this paper we will be especially interested in two types of graphs, namely lines and cycles.

\(^2\)If \( v, v' \) are not connected then \( d(v, v') = \infty \), however we only consider connected graphs in this paper.

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**Line graphs.** A line graph with \( k + 1 \) vertices is denoted by \( L_k = \{0, 1, \ldots, k\} \). We refer to an increase of the index as a movement in the right direction and similarly we refer to a decrease as a movement in the left direction. Clearly, in line graphs, every two vertices are connected by a single path. For \( v > v' \), we denote by the closed interval \([v, v']\) the set of vertices \( \{v, v + 1, \ldots, v' - 1, v'\} \), and by the open interval \((v, v')\) the set of vertices \( \{v + 1, \ldots, v' - 1\} \).

**Cycle graphs.** A cycle graph with \( k \) vertices is denoted by \( R_k = \{0, 1, \ldots, k - 1\} \). We refer to an increase (respectively, decrease) of the index as a movement in the clockwise (resp., counter-clockwise) direction. We denote the closed arc between \( v \) and \( v' \) in the clockwise direction as \([v, v']\) and the open arc in the clockwise direction as \((v, v')\).

In our model, there are \( n \) agents that are located on vertices of the graph. Let \( N = \{1, \ldots, n\} \) be the set of agents, and \( a = (a_1, \ldots, a_n) \in V^n \) be a location profile, where \( a_j \) denotes the location of agent \( j \) for every \( j \in N \). The locations of all the agents excluding agent \( j \) is denoted by \( a_{-j} \).

A facility location mechanism (or mechanism in short) for a graph \( G = (V, E) \) is a function \( f : V^n \to V \), specifying the chosen facility location for every location profile. Note that we assume here that the possible facility locations are exactly the vertices of the graphs. Given an agent \( j \)'s location \( a_j \in V \) and a facility location \( x \in V \), agent \( j \)'s cost is given by \( d(a_j, x) \). It is assumed that agents prefer to minimize their cost; that is, agents prefer having the facility located as close to them as possible (and are indifferent between locations of the same distance from them).

Interestingly, facility location mechanisms on the \( k \)-dimensional binary cube \( \{0, 1\}^k \) (or, more accurately, on certain subsets of it) are closely related to mechanisms in other domains, such as judgment aggregation and classification. We shall elaborate on this important topic in Sections 6 and 7.

### 2.1. Properties of mechanisms

We start with several definitions of mechanism properties, which are independent of the graph topology. While some of these properties are standard in the literature, we provide their definitions for completeness.

**Definition 2.1.** A mechanism \( f \) is **onto**, if for every \( x \in V \) there is \( a \in V^n \) s.t. \( f(a) = x \).

This property is a very basic requirement (sometimes referred to as society sovereignty), and as such we will restrict attention to mechanisms satisfying this condition. As can be seen below it is also the corollary of other natural properties. We are also interested in the following properties, which are stronger.

**Definition 2.2.** A mechanism \( f \) is **unanimous** if for every \( x \in V \), \( f(x, x, \ldots, x) = x \).

**Definition 2.3.** A location \( y \in V \) is said to Pareto dominate a location \( x \in V \) under a given profile if all the agents strictly prefer \( y \) over \( x \) (i.e. \( d(y, a_j) < d(x, a_j) \) for every \( j \in N \)). A mechanism \( f \) is **Pareto** if for all \( a \in V^n \), there is no location \( y \in V \) that Pareto dominates \( f(a) \) w.r.t. the profile \( a \).

Note that this requirement is slightly weaker than the more common definition of Pareto, requiring that no other location can strictly benefit one of the agents without hurting any other agent. It is easy to verify that Pareto implies unanimity, which in turn implies onto.

An agent \( j \) is said to be a **dictator** in \( f \) if for every location profile \( a \in V^n \), it holds that \( f(a) = a_j \). We define the following relaxation of the dictatorship notion.
In addition to the characterization of SP mechanisms, we shall be also interested in the performance of a given mechanism, as evaluated with respect to some well-defined objective function. The social cost function considered in this work is the sum of the distances of the agents’ locations from the chosen facility location. That is, given a profile \( a \) and every permutation of agents \( \pi : N \rightarrow N \), it holds that \( f(a_1, \ldots, a_n) = f(a_{\pi(1)}, \ldots, a_{\pi(n)}) \).

Note that a 0-dictator is essentially a dictator. It is argued that dictatorial mechanisms are “unfair” in the sense that the agent’s identity plays a major role in the decision of the facility location. Completely fair mechanisms that ignore agents’ identities altogether are said to be anonymous.

**Definition 2.5.** A mechanism \( f \) is anonymous, if for every profile \( a \) and every permutation of agents \( \pi : N \rightarrow N \), it holds that \( f(a_1, \ldots, a_n) = f(a_{\pi(1)}, \ldots, a_{\pi(n)}) \).

Our main interest is in strategyproof mechanisms, defined as follows.

**Definition 2.6.** A mechanism \( f \) is said to be strategyproof (SP), if no agent can strictly benefit by misreporting her location; that is, for every profile \( a \in V^n \), every agent \( j \in N \) and every alternative location \( a'_j \in V \), it holds that

\[
d(a_j, f(a'_j, a_{-j})) \geq d(a_j, f(a)).
\]

The following folk lemma gives a necessary condition for a mechanism to be onto and SP. For a proof see e.g. [Barberà and Peleg 1990].

**Lemma 2.7.** Every mechanism that is both onto and SP, is unanimous.

### 2.2. The social cost

In addition to the characterization of SP mechanisms, we shall be also interested in the performance of a given mechanism, as evaluated with respect to some well-defined objective function. The social cost function considered in this work is the sum of the distances of the agents’ locations from the chosen facility location. That is, given a location profile \( a \) and a facility location \( x \), the social cost is given by \( SC(x, a) = \sum_{j \in N} d(a_j, x) \).

When evaluating a mechanism’s performance, we use the standard worst-case approximation notion. Formally, given a profile \( a \), let \( \text{opt}(a) \) be an optimal facility location (i.e., \( \text{opt}(a) \in \arg\min_{x \in V} SC(x, a) \)). We say that a mechanism \( f \) provides an \( \alpha \)-approximation if for every \( a \in V^n \), \( SC(f(a), a) \leq \alpha \cdot SC(\text{opt}(a), a) \).

It is well known that in some domains, strategyproofness comes at the expense of the performance. A natural challenge is to identify cases where good performance can be achieved by a strategyproof mechanism.

As mentioned above, we assume that possible agent locations and facility locations coincide. In the more general case, the set of allowed facility locations (i.e., the range of \( f \)) may be more restricted than the set of possible agent locations \( V \). For example, a bus stop may need to be located on a main road, while the agents can be located anywhere in the city. Clearly, the necessary conditions for strategyproofness provided in this paper, as well as lower bounds on the approximation ratio, carry over to the general case.

### 3. SP Mechanism Over a Discrete Line

In this section we provide a characterization of onto SP mechanisms on a discrete line.

Given a location \( x \in L_k \), agent \( j \)'s cost is \( d(a_j, x) = |a_j - x| \). Given a set of vertices \( S \subseteq V \), we denote by \(|S|\) the cardinality of \( S \). In the following definitions, \( a_j, b_j \) etc. are possible locations in \( L_k \) for agent \( j \).

**Definition 3.1.** A mechanism \( f \) on a line is monotone (MON) if for every \( j \in N \) and every \( b_j > a_j \), \( f(a, b_j) \geq f(a_{-j}, a_j) \).
In other words, monotonicity of a mechanism means that if an agent moves in a certain direction, the outcome cannot move in the other direction as an effect. The following two properties bound the effect of an agent’s movement on the outcome of the mechanism.

**Definition 3.2.** A mechanism \( f \) is \( m \)-step independent (\( m \)-SI) if the two following properties hold: (a) For every \( j \in N, a'_j > a_j \), if \( d([a_j, a'_j], f(a)) > m \), then \( f(a'_j, a_{-j}) = f(a) \). (b) For every \( j \in N, a'_j \leq a_j \), if \( d([a'_j, a_j], f(a)) > m \), then \( f(a'_j, a_{-j}) = f(a) \).

**Definition 3.3.** A mechanism \( f \) is disjoint independent (DI) if for every \( j \in N, a'_j \in L_k \), if \( f(a) = x \neq x' = f(a'_j, a_{-j}) \), then \( |A \cap X| \geq 2 \), where \( A \) is the segment defined by \( a_j \) and \( a'_j \) (i.e., \( A = [\min_j (a_j, a'_j), \max_j (a_j, a'_j)] \)) and \( X \) is the segment defined by \( x, x' \).

Intuitively, \( m \)-SI means that a deviation that occurs in an interval sufficiently far from the original outcome does not affect it. The DI property means that an agent can affect the outcome of the mechanism only in a way in which its trajectory intersects the trajectory of the facility in at least two consecutive points.

A mechanism is said to be strongly \( m \)-step independent (\( m \)-SSI) if it is both \( m \)-SI and DI. For example, the median mechanism (and in fact any order statistics mechanism) is strongly \( 0 \)-SSI.

Our first primary result characterizes all the mechanisms that satisfy the requirements of onto and SP on the line.

**Theorem 3.4.** An onto mechanism \( f \) on the line is SP if and only if it is MON and 1-SSI.

In the remainder of this section we sketch the proof of Theorem 3.4. Full proofs of all lemmas are deferred to Appendix A. An alternative characterization is given in Section 6, using the notations of the binary cube.

**Lemma 3.5.** Every SP mechanism is monotone.

**Lemma 3.6.** A monotone mechanism \( f \) is Pareto iff it is unanimous.

Notice that the Pareto property (Def. 2.3) has a simpler form in this domain: \( f(a) \in [\min_{j \in N} a_j, \max_{j \in N} a_j] \). The following lemma is the main building block in the proof of Theorem 3.4.

**Lemma 3.7.** Every SP, unanimous mechanism for the line is 1-SI.

A few remarks are in order. It is not hard to verify (see Lemma A.1 in the appendix) that every 0-SI monotone mechanism on the line is SP. This lemma can be seen as a particular case of Nehring and Puppe (2007) theorem. They show that for any subset of the binary cube (see Section 3), 0-SI (called IIA) and monotone are sufficient and necessary conditions for being an SP mechanism, for a certain definition of SP that is stronger than ours. The following example shows that 0-SI is not a necessary condition: Consider a setting with two players and the following mechanism \( f \) on \( L_2 \): \( f(a_1, a_2) = 2 \) if \( a_1 = 2 \) or \( a_2 = 2 \), \( f(a_1, a_2) = 1 \) if \( a_1 = a_2 = 1 \), and \( f(a_1, a_2) = 0 \) otherwise. The reader can check that this is an SP, onto and unanimous mechanism; however, it is not 0-SI, since moving from the profile \((0, 1)\) to \((0, 2)\) changes the result from 0 to 2.

We now turn to sketch the proof of the main theorem of this section.

**Proof of Theorem 3.4.** Suppose \( f \) is an onto SP mechanism; then, by Lemmas 2.7 and 3.5, it is also monotone and unanimous, and therefore, by Lemma 3.7, it is 1-SI. Suppose that \( f \) does not satisfy 1-SSI; then, there is an agent \( i \) that violates DI (i.e., caused the violation). Therefore, there is a profile \((a_i, a_{-i})\) and deviation \( a'_i \) s.t.
\[ f(a_i, a_{-i}) = x = x' = f(a'_i, a_{-i}) \] but \(|A \cap X| \in \{0, 1\}\) (\(A\) and \(X\) are the segments as used in Def.3.3). W.l.o.g. assume \(a_i < a'_i\), \(f\) satisfies 1-SI and hence \(d([a_i, a'_i]) < 2\). It is easy to see that \(x = a_i + 1\) and \(i\) benefits by this move away from the facility location (since, by monotonicity, the facility moves in the same direction).

We now prove the other direction. Suppose \(f\) is an onto, monotone and 1-SSI mechanism. We will show that \(f\) is also SP. Suppose some agent \(j\) moves from \(a_j\) to \(a'_j > a_j\) and by that causes the facility to move from \(x = f(a)\) to \(x' = f(a'_j, a_{-j})\)(the proof for movement to the left is symmetric). By monotonicity, \(x' \geq x\). If \(x \geq a_j\), then agent \(j\) does not benefit from the deviation. Otherwise, \(x < a_j\); then, by 1-SI it holds that \(x = a_j - 1\) (otherwise the facility will not move). By DI, it must hold that \(|[a_j, a'_j] \cap [x, x']| \geq 2\), which means that \(x' \geq a_j + 1\). Here again, agent \(j\) does not benefit from the deviation.

3.1. Descriptive and axiomatic characterizations

For continuous lines, the set of SP and onto mechanisms has been characterized as all generalized median voting schemes (g.m.v.s) [Border and Jordan 1983; Schummer and Vohra 2004]. This basically means that \(f(a)\) is the median selection from some subset of agents. By slightly modifying our definitions above (informally, by replacing the 1-SI requirement with a 0-SI requirement), we get an alternative, axiomatic characterization that is similar to the one we give for the discrete case. While this definition seems very different from the definition of a g.m.v.s., the two definitions coincide by Theorem 3.4 (as both are equivalent to requiring onto and SP). Similarly, it is possible to give a descriptive characterization in the spirit of g.m.v.s. in the discrete case.

4. SP MECHANISMS ON A DISCRETE CYCLE

In this section we move forward from line graphs, where many SP mechanisms exist, to cycle graphs. After inspecting the main differences from the continuous case, we will continue to our main result, which puts a strong limitation on the possible SP mechanisms.

Schummer and Vohra [2004] proved that any onto SP mechanism on the continuous cycle must be a dictatorship. However, this is not true for discrete cycles. Clearly, any dictator mechanism is both unanimous and SP, but the converse does not hold.

Consider some cycle \(R_k\) of even length \(k\), with any number of agents. The following is an example of an SP mechanism: The cycle is partitioned to \(k/2\) pairs of neighboring points. First, the pair in which agent 1 resides is chosen. The location within this pair of points is decided by a majority vote of all other \(n - 1\) agents. This is not a dictatorial mechanism, and in fact every agent has some small effect on the outcome in some profiles.

Moreover, if the cycle contains only few vertices, then there are even completely anonymous mechanisms (i.e., very far from dictatorships) that are SP. See Section 4.4 for detailed examples.

We still want to claim that when \(k\) is large enough, then any onto SP mechanism on the cycle \(R_k\) is “close” to a dictator. Note that even in the example above, the facility is always next to agent 1, which makes him a 1-dictator. The main result of this section shows that this is always the case (see formal statement in Theorem 4.10).

Main theorem. For sufficiently large cycles, any onto SP mechanism is 1-dictatorial.

In Section 5 we complete the characterization (for even \(k\)) by considering the embedding of the cycle in the binary cube.
As a proof outline of the main theorem, we go through the following steps. We first consider the case of two agents, proving that any SP mechanism must be Pareto and then that the facility must always be next to one of the agents. It then follows that a 1-dictator must exist. The next step is proving the same for three agents, using a reduction to the \( n = 2 \) case. Finally, we extend the result to any number of agents using an inductive argument close to the one used by Schummer and Vohra [2004] (and to similar ideas in [Kalai and Muller 1977; Svensson 1999]).

Before diving into the case of 2 agents, we prove two general lemmas for onto SP mechanisms. (a more formal statement is in Appendix [B]. Let a, a’ be two profiles that differ only by the location of one agent (w.l.o.g. agent 1), and denote \( x = f(a), x' = f(a') \). We refer to it as if agent 1 moves from \( a_1 \) to \( a_1' \).

**Lemma 4.1.** If agent 1 moves closer to \( x \) along the shorter arc between them, then \( x' = x \). I.e., if \( |(a, x)| \leq \lfloor k/2 \rfloor \) and \( a' \in (a, x) \) then \( x' = x \).

**Proof.** W.l.o.g. we assume that a moves clockwise. First assume that she moves one step \( a' = a + 1 \). Assume, toward a contradiction, that \( y = f(a', a_{a-1}) \neq x \). Then either \( y \in [a, x) \) (in which case \( a \rightarrow a' \) is a manipulation), or \( y \in (x, a] \). If \( |(a, y)| \leq \lfloor k/2 \rfloor \), then since \( x \in (a', y) \) it is closer to \( a' = a + 1 \) than \( y \), meaning that \( a \rightarrow a' \) is a manipulation.

Therefore, the shorter arc between \( a, y \) is \( |y, a| \), of length \( \leq \lfloor k/2 \rfloor \). Of course, \( d(y, a) \geq d(x, a) \) (otherwise \( a \rightarrow a' \) is a manipulation). However, this means that

\[
d(a', x) = d(a, x) - 1 \leq d(a, y) - 1 = d(a', y) - 1 < d(a', y),
\]

i.e., that \( a' \rightarrow a \) is a manipulation.

Therefore, we proved that one step toward \( x \) does not move the facility. By induction on the number of steps needed to move from \( a \) to \( a' \), we get the lemma. \( \square \)

**Lemma 4.2.** Suppose that agent 1 moves one step away from \( x \) (along the longer arc between \( x \) and \( x' \)). Let \( y \) be the point on the longer arc s.t. \( d(a', y) = d(a, x) \). Then either \( x' = x \) (no change); or \( d(x', y) \leq 1 \). (If \( x \) is antipodal to \( a \), then trivially \( a \) cannot move it.)

**Proof.** W.l.o.g. \( a' = a + 1 \). If \( x' \in [x, y - 2] \) then \( a \rightarrow a' \) is a manipulation. If \( x' \in [y + 2, x) \) then \( d(a', x') > d(a', x) \), and thus \( a \rightarrow a' \) is a manipulation. \( \square \)

For the case of the cycle we define a specific property of cycle-Pareto. For the exact definition see Def. [B.1] in the appendix. However, for our proof sketch it is sufficient to note that cycle-Pareto is very similar to Def. [2.3] (in fact for even size cycles the definitions coincide).

We prove below that any SP mechanism for 2 agents on large enough cycles must satisfy cycle-Pareto. However, as Lemma [B.3] in the appendix shows, this result is true for any number of agents.

### 4.1. Two agents on the cycle

**Lemma 4.3.** If \( a, b, f(a, b) \) are on the same semi-cycle, then \( f(a, b) \) is between \( a, b \).

**Proof.** Assume otherwise, w.l.o.g. \( a \in (b, f(a, b)) \). Then \( b \) can manipulate by reporting \( a \), since \( f(a, a) \) is closer to \( b \) than \( f(a, b) \). \( \square \)

**Lemma 4.4.** Let \( k \geq 13, n = 2 \). If \( f \) is SP and onto on \( R_k \), then \( f \) is cycle-Pareto.

We give a simpler proof for large cycles. The full proof appears in Appendix [B.1].

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3I.e., there is a segment of length at most \( \frac{k}{2} \) that includes the three points.
Finally, denote $z$. We show that

From a similar argument, we get that $|x, x|$. Then we get a contradiction as $x' = f(a', b'), x'' = f(a'', b'')$. See Figure 1 for an illustration.

We show that $b'$ must be roughly antipodal to $a'$, as otherwise we get a violating profile that is contradicting (1). Then by Lemma 4.2 $x'$ is a reflection of $x$ along the axis $a' \leftrightarrow b'$. It follows that $||x', x|| = ||x', b'|| + ||b', x|| = 2||b', x|| + 3 = k - 40 \pm 5$.

Finally, denote $z = f(a'', b')$. By Lemma 4.1 $z = f(a'', b'') = x''$, as agent 2 approaches $x''$. On the other hand, by the same argument $z = f(a', b') = x'$, as agent 1 approaches $x'$. Then we get a contradiction as $x' = z = x''$. □

**Lemma 4.5.** Let $k \geq 13$. For all $a, b \in R_k$, $x = f(a, b)$, $d(a, x) \leq 1$ or $d(b, x) \leq 1$.

**Proof.** Assume that there is some violating profile, then w.l.o.g. it is $x = f(a, b)$, where $x = b + 2$, and $a > x$. Also, by cycle-Pareto and Lemma 4.1 $a' = b + 5$ and $a'' = b + 7$ have the same outcome $x = f(a', b) = f(a'', b)$.

However, we show in the (full) proof of Lemma 4.4 that exactly this pair of profiles leads to a contradiction. □

We can now use the results above to prove the main result for two agents.

**Theorem 4.6.** Assume $k \geq 13$, $n = 2$. Let $f$ be an onto SP mechanism on $R_k$, then $f$ is a 1-dictator.

**Proof.** Take some profile $a, b$ where $x = f(a, b)$, $d(x, b) > 1$. By Lemma 4.5 $x$ is near $a$, i.e., $d(a, x) \leq 1$. We will show that agent 1 is a 1-dictator (in the symmetric case, agent 2 will be a 1-dictator). Assume, toward a contradiction, that there is some location $b'$ for agent 2 s.t. $d(y, a) > 1$, where $y = f(a, b')$ (and by Lemma 4.5 $d(y, b') \leq 1$). We can gradually move agent 2 from $b$ to $b'$ until the change occurs, and thus w.l.o.g.
\( b' = b + 1 \). By Lemma 4.1, moving agent 2 toward \( x \) cannot change the outcome, thus on the arc \( [x, b'] \) the order is \( x < x + 1 < b < b' \).

We must have \( d(b, x) \leq d(b, y) \), as otherwise there is a manipulation \( b \rightarrow b' \). Thus \( d(b, a) - 1 \leq d(b, b') + 1 = 2 \), i.e., \( d(a, b) \leq 3 \). Also, \( d(a, b) \geq 1 \) since otherwise \( d(y, a) = d(y, b) \leq 1 \) in contradiction to our assumption. Thus there are three possible cases, and we will show that each leads a contradiction.

(I) If \( d(a, b) = 1 \), then since \( d(x, b) > 1 \) we have \( x = a - 1 \) (in contradiction to Lemma 4.3).

(II) If \( d(a, b) = 2 \), then \( x = a \) (since \( x = a - 1 \) contradicts lemma 4.3). Thus \( d(y, b) \geq d(x, b) = d(a, b) = 2 \) which means \( y = b' + 1 = b + 2 \). This induces a manipulation for agent 1 \( a \rightarrow b' \) (by unanimity).

(III) If \( d(a, b) = 3 \), then since \( k > 8 \), all of the points are on a semi-cycle and thus \( x \in \{a, a + 1\} \), \( y \in \{b', b' - 1\} \) (again, by lemma 4.3). However clearly this means that \( d(y, b) \leq 1 < d(x, b) \), and there is a manipulation \( b \rightarrow b' \) for agent 2. □

4.2. Three agents on the cycle

**Lemma 4.7.** Assume \( k \geq 13 \), \( n = 3 \). Let \( f \) be a unanimous SP mechanism on \( R_k \). Then either \( f \) has a 1-dictator, or any pair is a 1-dictator. That is if there are two agents \( j, j' \) s.t. \( a_j = a_{j'} \), then \( d(f(a), a_j) \leq 1 \).

**Proof.** Let \( f \) be an SP unanimous rule for \( n = 3 \) agents. We define a two agent mechanism for every pair \( j, j' \) s.t. \( j, j' \in N \) by letting \( j \) be a duplicate of \( j' \) (For ease of notation we’ll refer to the agents of \( g^{j, j'} \) by agent I and agent II, the third agent by \( j'' \), and the original agents by agent 1, agent 2, and agent 3 ),

\[
g^{12}(a, b) = f(a, a, b); \quad g^{23}(a, b) = f(b, a, a); \quad g^{31}(a, b) = f(a, b, a).
\]

Clearly, the mechanism \( g^{j, j'} \) is unanimous, since \( g^{j, j'}(a, a) = f(a, a, a) = a \) for all \( j, j' \).

We argue that \( g^{j, j'} \) is SP. Indeed, otherwise there is a manipulation either for agent II (which is also a manipulation in \( f \), which is a contradiction to SP) or for agent I (say, \( a \rightarrow a' \)). In the latter case we can construct a manipulation in \( f \) by iteratively switching agents \( j, j' \) from \( a \) to \( a' \). Either \( j \) or \( j' \) strictly gains by this move and thus has a manipulation.

Since \( g^{j, j'} \) is a unanimous and SP, by Theorem 4.6 it has a 1-dictator. If the dictator is agent II then \( j'' \) is also a 1-dictator of \( f \). Otherwise, suppose that \( f(a_j, a_j', a_j'') = x \), and \( d(x, a_{j''}) > 1 \). However It the follows by Lemma 4.1 that \( f(x, a_{j''}) = x \) as well, which is a contradiction.

If agent I is a 1-dictator of \( g \), then whenever \( a_j = a_{j'} \), \( d(f(a), a_j) \leq 1 \). □

**Lemma 4.8.** Let \( f \) be an SP, unanimous rule for 3 agents on \( R_k \) for \( k \geq 13 \). For all \( a, b, c \in R_k \), \( x = f(a, b, c) \), \( d(a, x) \leq 1 \) or \( d(b, x) \leq 1 \) or \( d(c, x) \leq 1 \).

**Proof.** By Lemma 4.7 either there is a 1-dictator (in which case we are done), or every pair of agents standing together serve as a 1-dictator.

Let \( u_1, u_2, u_3, x = f(u_1, u_2, u_3) \) s.t. \( x \) is at least 2 steps from all agents. We have that there is a semi-cycle in which \( x \) and two other points are consequent, and thus \( x \) must be between them (otherwise the more distant agent of the two has a manipulation, similarly to Lemma 4.3). W.l.o.g. \( u_1 + 1 < u_2 < u_2 + 1 \) (i.e., ordered that way on an arc). Now suppose that agent 3 moves to \( u_1 \) or \( u_2 \), whichever closer to her (assume \( u_1 \)). Then \( y = f(u_1, u_2, u_1) \) is close to \( u_1 \). We thus have

\[
d(u_3, y) \leq d(u_3, u_1) + d(u_1, y) \leq (d(u_3, x) - 2) + 1 < d(u_3, x),
\]

i.e., there is a manipulation for agent 3. □
THEOREM 4.9. Assume \( k \geq 22, n = 3 \). Let \( f \) be an onto SP mechanism on \( R_k \), then \( f \) is a 1-dictator.

We give a simpler proof for large cycles. The full proof appears in Appendix B.2.

PROOF FOR \( k \geq 100 \). Assume, toward a contradiction, that \( f \) has no 1-dictator. By Lemmas 4.4.7, we know that \( f(a) \) is always close to at least one agent, and if there is a pair in the same place \( a^* \) then \( d(f(a), a^*) \leq 1 \).

Let \( a \) be a profile where \( a_2 = a_1 - 20; a_3 = a_1 + 20 \). Thus all three agents and \( x = f(a) \) are on the same semi-cycle and \( x \) is near \( a_1 \) (otherwise there is a manipulation for agent 2 or agent 3 by joining agent 1). Let \( a_2' = a_2 + 8 \), then by Lemma 4.11 \( f(a_1, a_2', a_3) = x \). From each profile \( a, a' \) we move agent 3 toward \( a_2 \) (or \( a_2' \)) along the longer arc between them, until the facility “jumps” to agent 2. This must occur at some point by Lemma 4.11. Denote by \( b_3 \) [resp., \( b_3' \)] the first point s.t. \( f(a_1, a_2, b_3) \neq f(a) \) [resp., \( f(a_1, a_2', b_3') \neq f(a') \)]. It must hold that \( b_3 \) is in the middle of the long arc between \( a_1, a_2 \) (plus or minus 1), since otherwise there would be a manipulation \( b_3 \to b_3-1 \) or vice versa. Thus \( b_3 \in [a_1 + k/2 - 11, a_1 + k/2 - 9] \). From the same argument, \( b_3' \in [a_1 + k/2 - 7, a_1 + k/2 - 5] \) and therefore \( b_3' > b_3 \). Finally, consider the two profiles \( z = f(a_1, a_2', b_3); w = f(a_1, a_2, b_3) \). Since \( b_3 < b_3' \), \( z \) is next to \( a_1 \), and thus \( d(z, a_2') \geq d(a_1, a_2') - 1 = 11 \). On the other hand, \( w \) is next to \( a_2 \) (by the definition of \( b_3 \)), thus \( d(w, a_2') \leq d(a_2', a_2) + 1 = 9 \leq d(z, a_2') \), which means that \( a_2' \to a_2 \) is a manipulation for agent 2, in contradiction to SP.

4.3. \( n \) agents on the cycle

Finally, we leverage the results of the previous sections to obtain a necessary condition for mechanisms on the discrete cycle for the general case of \( n \) agents.

THEOREM 4.10. Let \( f \) be an onto and SP mechanism on \( R_k \), where \( k \geq 22 \), then \( f \) is 1-dictatorial.

PROOF. We assume by induction for every \( m < n \) (we know it holds for \( n \leq 3 \)). Let \( f \) be an SP unanimous rule for \( n \geq 4 \) agents. We define two mechanisms for \( n-1 \) agents:

\[
g(a_{n-1}) = f(a_1, a_2, a_{n-1}) \quad ; \quad h(a_{n-3}) = f(a_3, a_4, a_{n-3}).
\]

Now, similarly to the proof of Lemma 4.11, both \( g, h \) are unanimous and SP and therefore both are 1-dictator mechanisms. If we have that some \( j \neq 2 \) is the dictator of \( g \) we are done (since then \( j \) is a 1-dictator of \( f \)), and similarly for any \( j' \neq 4 \) in \( h \).

Assume, toward a contradiction that agents 2 and 4 are the 1-dictators of \( g, h \), respectively. Then take any profile where \( a_1 = a_2, a_3 = a_4 \) and \( d(a_2, a_4) > 2 \) (this is always possible for \( k > 4 \)). We then have that \( x = f(a) \) holds both \( d(x, a_2) \leq 1 \) and \( d(x, a_4) \leq 1 \), i.e., \( d(a_2, a_4) \leq 2 \) in contradiction to the way we defined the profile.

4.4. Small cycles

A natural question is the critical size of a cycle, for which there still exist SP mechanisms that are not 1-dictatorial. The proofs above show that the critical size for \( n = 2 \) is at most 12, and for \( n \geq 3 \) it is at most 21. We want to know whether these bounds are tight.

PROPOSITION 4.11. There are onto and anonymous SP mechanisms for two agents on \( R_k \), for all \( k \leq 12 \).

PROPOSITION 4.12. There are onto and anonymous SP mechanisms for three agents on \( R_k \), for all \( k \leq 14 \) and \( k = 16 \).
For \( k \leq 7 \), the following “median-like” mechanism will work for \( n = 3 \): let \((a_3, a_1)\) be the longest clockwise arc between agents, then \( f(a) = a_2 \). Break ties clockwise, if needed. For two agents we simply fix the location of one virtual agent. (see proof in Appendix B.3).

For higher values of \( k \) the “median” mechanism is no longer SP, but we have been able to construct anonymous SP mechanisms using a computer search for all the specified values. A tabular description of these mechanisms is available online [A].

Proposition 4.11 settles the question of the maximal size for which non-1-dictatorial mechanisms for two agents exist. For three agents, we close the gap between Proposition 4.12 and Theorem 4.9 by performing an exhaustive search on all mechanisms with three agents for \( k \in \{15, 17, 18, 19, 20, 21\} \). Indeed, it turns out that every mechanism in this range must be 1-dictatorial. Thus we have a full characterization of the cycle sizes for which non-1-dictatorial SP mechanisms exist.

As a direct corollary from Proposition 4.12 we get the following result, by adding any number of agents from which the mechanism ignores.

**Proposition 4.13.** For all \( n \geq 3 \), \( k \leq 14 \) and \( k = 16 \), there are onto SP mechanisms for \( n \) agents on \( R_k \) that treat the first 3 agents symmetrically. In particular, these mechanisms are not 1-dictatorial.

Note however that the resulting mechanism is not an anonymous one.

### 5. IMPLICATIONS OF THE MAIN THEOREM

In this section we cover some strong implications of the result that any SP mechanism on a large cycle must be almost-dictatorial.

#### 5.1. Cyclic graphs

The first implication is that this result extends to a much larger family of graphs. A natural conjecture is that any SP (and onto) mechanism on any graph containing a cycle that matches the conditions of the theorem, must be 1-dictatorial on a subdomain. However, we need to be careful. In the continuous case studied by Schummer and Vohra [2004], any cyclic graph contains a continuous cycle and thus their negative result automatically applies.

In the discrete case, this is only guaranteed to be true if we add edges outside the cycle. We define a **minimal cycle** as a cycle that is not cut by any string. Equivalently, the shortest path between every two vertices on the cycle is going through the edges of the cycles. The extension of our main theorem is as follows.

**Corollary 5.1.** Let \( G = (V, E) \) be graph that contains some minimal cycle \( R \subseteq V \) that is sufficiently large (according to Table I). Then any SP onto mechanism on \( G \) has a “cycle 1-dictator” \( i \in N \). That is, if all agents lie on \( R \) then \( d(f(a), a_i) \leq 1 \).

**Proof Sketch.** Let \( f \) be an onto SP mechanism on \( G \). We argue that whenever \( a \in (R)^n \) (i.e. all agents are on the cycle \( R \)), then \( f(a) \in R \) as well. Assume otherwise, then by iteratively and gradually moving all agents to closest point on the cycle, the facility eventually moves to the cycle. Since moving the facility to a point between the agent and the original location is a manipulation, the facility must “jump” to a distant location (from the agent) on the cycle. Moreover, there must be at least two such distinct locations, induced by agents moving clockwise (a set \( S \subseteq N \)) and agents moving counterclockwise (\( T \subseteq N \)). Moving from one such profile to the other requires a

---

4While the number of mechanisms for 3 agents and bounded \( k \) is finite, the size of the search space is huge \((k^{3n}(k^n))\). Thus any naive search would be infeasible. However, by using the lemmas from Section 4 we can significantly reduce the search space so that the search completes in several minutes.
single step by two different agents, one from each set. One of these steps must move the facility from a point closer to $S$ to a point closer to $T$, and therefore it is a manipulation for one of the agents. \qed

If we take a large cycle and add internal edges (so that it is no longer minimal), then there may be non-dictatorial mechanisms that are SP. As a simple example, the main theorem applies on $R_{14}$ with $n = 2$. However if we add the edge $(0, 7)$, this forms two cycles of length 8. The following mechanism is SP and onto: if the two agents are on different cycles, then $f(a) = 0$. If they are on the same cycle, then we apply the “median-like” mechanism for $R_8$ described in Appendix [B.3.3] where the point 0 serves as the dummy agent for both cycles.

5.2. The social cost

Dictatorial mechanisms typically have poor performance in terms of social welfare (or cost). While for a low number of agents a dictatorial facility location mechanism is not so grave (in fact, for $n = 2$ the dictator mechanism is optimal w.r.t. the social cost), for more agents the main theorem provides us with a lower bound that linearly increases with the number of agents (the corollary still holds for lower values of $k$, as appear on Table [1]).

**Corollary 5.2.** Every SP mechanism on $R_k$ for $k \geq 22$ has an approximation ratio of at least $\frac{3}{5}n$. The ratio converges to $n - 1$ as $k$ tends to infinity.

**Proof.** If the mechanism is not unanimous, it has an infinite approximation ratio. Otherwise it is a 1-dictator, w.l.o.g. agent $n$ is the 1-dictator. Let $a_1 = a_2 = \ldots = a_{n-1} = k$, and $a_n = \lfloor \frac{k}{2} \rfloor$. Clearly, the optimal location is $\text{opt} = a_1$, and the optimal total distance from all agents is $\lfloor \frac{k}{2} \rfloor$. However, $f(a) = |\frac{k}{2}| ± 1$, and the total distance from the agents is at least $n - 1 \left( \frac{k}{2} - 1 \right)$ (in fact $\min\{((n - 1)|\frac{k}{2}|, n \left( \frac{k}{2} - 1 \right)\})$. Thus the approximation ratio for $k \geq 22$ is

$$\frac{\text{SC}(f(a))}{\text{SC}(\text{opt}(a))} \geq \left(n - 1\right) \frac{\left|\frac{k}{2}\right| - 1}{\left|\frac{k}{2}\right|} \geq \frac{2n}{9} $$

thereby proving the assertion. \qed

5.3. The continuous case

As we mentioned in Section [3.1] for continuous lines, one can repeat the steps of our proof, with some adjustments, when the underlying graph is continuous. This results in an alternative proof that every onto SP mechanism on a continuous cycle is dictatorial. In fact, some steps of our proof are greatly simplified in the continuous case, leaving us with a relatively short and intuitive proof for Theorem 2 in [Schummer and Vohra, 2004] (p.22). We do not include the full details here.

6. THE BINARY CUBE

A binary cube of dimension $k$ is denoted by $C_k$. The set of vertices in $C_k$ is the set of binary vectors of size $k$. Two vertices $v, v' \in C_k$ are connected if their hamming distance (i.e., the number of coordinates in which they differ) is 1. Given a vertex $v$, we denote by $v[i] \in \{0, 1\}$ the $i$'th coordinate of $v$. Therefore, $d(v, v') = \{|i : v[i] \neq v'[i]\}|$.

We next define several properties of mechanisms for the binary cube $C_k$. These definitions will serve several purposes: first, by considering a natural embedding of $R_k$ in $C_k$ we can provide a full characterization of SP mechanisms on the cycle in terms of the cube dimensions. Interestingly, we give an alternative characterization for mechanisms on the line using the same properties. Second, we consider some implications of our results on other domains, which correspond to the binary cube.
Suppose that \( V \) is some subset of \( C_k \). Since every location can be thought of as having \( k \) coordinates (or attributes), the cube structure calls for some new definitions.

**Definition 6.1.** A mechanism \( f \) is Cube-monotone, if changing coordinate \( i \) of an agent can only change coordinate \( i \) in the same direction. That is, if \( a_j[i] \neq a'_j[i] \) and \( f(a)[i] \neq f(a_j, a'_j)[i] \), then \( f(a)[i] = a_j[i] \).

Another property often considered in a multi-attribute setting is independence in irrelevant attributes. This means that coordinate \( i \) of the facility is only determined by the values of coordinate \( i \) of the agents’ locations. While this property seems unnatural in the general case of aggregating agent location on a subset of the cube, it is reasonable in a lot of related aggregation problems. For example, in preference aggregation the IIA property means pair-wise aggregation and is accepted as a desired property. As was shown by Dietrich and List [2007a] preference aggregation can be seen as aggregation on the cube. We relax this notion as follows.

**Definition 6.2.** A mechanism \( f \) is \( m \)-independent of irrelevant attributes (\( m \)-IIA) if \( f(a)[i] \) is determined by coordinates \( i - m, \ldots, i + m \) of the voters in \( a \).

Note that the \( m \)-IIA property depends on coordinates order, and is not preserved under a permutation of coordinates’ names. 0-IIA is just IIA. The following property is also quite natural.

**Definition 6.3.** A mechanism \( f \) is independent of disjoint attributes (IDA), if the coordinates changed by the agent and the coordinates changed in the facility (if it moved) always intersect. Formally, if \( a_j, a'_j \) differ by coordinates \( S \subseteq K \), and \( f(a_j, a_{-j}), f(a'_j, a_{-j}) \) differ by coordinates \( T \subseteq K \), then either \( T = \emptyset \) (i.e. no change in outcome) or \( S \cap T \neq \emptyset \).

A similar property was suggested by Dietrich [2007] as independence in irrelevant information (in our case a coordinate is relevant to its neighborhood, and irrelevant to all other coordinates).

**Definition 6.4.** We say that a mechanism \( f \) is Cube-Pareto, if whenever all the agents agree on the same coordinate (vote the same), then this is the aggregated coordinate as well.

### 6.1. Embedding the line in the binary cube

We give a natural embedding of \( L_k \) in \( C_k \). Map every \( x \in L_k = \{0, 1, \ldots, k\} \) to a vector \( \varphi(x) \in (0, 1)^k \), whose first \( x \) entries are 1. Thus \( \varphi(x)[i] = 1 \) iff \( i \leq x \). It is easy to verify that \( \varphi \) is distance-preserving, i.e., that \( d(\varphi(x), \varphi(x')) = |x - x'| = d(x, x') \).

Every mechanism \( f \) on \( L_k \) induces a mechanism \( f_\varphi \) on the embedded space \( \varphi(L_k) \subseteq C_k \). The following correspondences of properties follow directly from distance preserving of the mapping \( \varphi \).

**Lemma 6.5.** Let \( f \) be a mechanism on \( L_k \).

1. \( f \) is monotone iff \( f_\varphi \) is Cube-monotone.
2. \( f \) is \( m \)-SI iff \( f_\varphi \) is \( m \)-IIA.
3. \( f \) is DI iff \( f_\varphi \) is IDA.
4. \( f \) is Pareto iff \( f_\varphi \) is Cube-Pareto.

By Lemma 6.5 we get the following theorem, which is equivalent to Theorem 3.4. It demonstrates that the same set of properties (defined w.r.t. the binary cube) is useful for characterizing mechanisms for both lines and cycles.
Fig. 2. A classification dataset in \( \mathbb{R}^2 \). The \( k \) data points of all three agents are identical, but the labels, i.e., their types, are different. The best linear classifier with respect to each agent is also shown (the arrow marks the positive half-space of the separator). Only the rightmost dataset is realizable.

**Theorem 6.6.** An onto mechanism on \( \varphi(L_k) \) (The line embedded in \( C_k \)) is SP if and only if it is 1-IIA, Cube-monotone, and IDA.

### 6.2. Full characterization of SP mechanism on the cycle

Every cycle of even length can be thought of as “two lines attached in their ends”. Indeed, \( R_{2k} \) can be embedded in the binary cube \( C_k \) in a very similar way to the embedding of the line. This is by mapping the first \( k \) points on the cycle (setting order and orientation on the cycle. We later show that these can be arbitrarily chosen) to vectors of the form \( 0^k 1^k \) (as with \( L_k \)), and the remaining \( k \) points to vectors of the form \( 1^k 0^k \). In particular, \( \varphi(0) = 0^k \), and \( \varphi(k) = 1^k \). As with \( L_k \), it is not hard to verify that our mapping preserves distances, as

\[
d(\varphi(x), \varphi(x')) = d(x, x') = |x - x'| \pmod{2^k}.
\]

We can now turn to completing the characterization of SP mechanisms on the cycle, extending Theorem 4.10.

**Theorem 6.7.** Let \( 2k \geq 18 \) (or \( 2k \geq 14 \) for \( n = 2 \)). An onto mechanism on the cycle \( R_{2k} \) is SP if and only if it is 1-dictatorial, Cube-monotone, and IDA.

Notice that all these properties do not depend on the choice of embedding (from the \( 2k \) ways to choose starting point and direction). The coordinates can be seen as a geometric property telling us where is the facility w.r.t. the agents and their antipodal points, hence the properties are independent of the embedding. For instance, Cube-Pareto can be interpreted as - “for any semi-cycle s.t. all the agents are in this semi-cycle, the facility should lie in it as well”.

### 6.3. Beyond facility location - Implications on other domains

As mentioned in the introduction, there is a mapping between facility location mechanisms in our model, and binary classification mechanisms operating on realizable datasets. A natural question is whether we can derive characterization and approximation results that will apply for the classification setting, and in particular to natural concept classes that are in use in the machine learning literature.

We exemplify such a derivation for linear classifiers in \( \mathbb{R}^d \), which is one of the most prominent concept classes used in the machine learning framework. The connections between Facility location, Classification, and other mechanism design problems are studied in detail by Meir et al. [2012].

**Linear Classifiers.** A linear classifier is composed of a unit vector \( w \in \mathbb{R}^d \) and a scalar \( u \). It classifies every data point \( x \in \mathbb{R}^d \), to \( \{+, -\} \), according to \( \text{sign}(\langle w, x \rangle - u) \) (see Figure 2). A linear classifier in \( \mathbb{R}^1 \) is just a scalar \( u \) and a direction \( w \in \{-, +\} \).
In a binary classification problem, we are given a set of labeled data points \( S \), and are required to return a classifier \( c = \langle w, u \rangle \). The quality of the classifier \( c \) is measured according to the number of errors that \( c \) makes on \( S \), which we want to minimize. Formally, \( \text{err}(c, S) = |\{ (x, y) \in S : c(x) \neq y \}| \). The classifier with the lowest number of errors on \( S \) is denoted by \( \text{opt}(S) \).

A classification mechanism is a function \( M \) mapping every labeled dataset to a classifier. Typically, it is assumed that labels are acquired via some objective process (which may be noisy), in which case classification is just an optimization problem. The approximation ratio of a classification mechanism is the maximal ratio \( \frac{\text{err}(M(S), S)}{\text{err}(\text{opt}(S), S)} \), taken over all possible datasets \( S \).

However, in certain situations labels are reported by several self-interested agents, which may have different opinions on the appropriate label for each data point. In such cases, every classification mechanism induces a game, in which agents may lie in order to bias the outcome classifier closer to their own view, similarly to false reports in the facility location problem. Therefore, we are interested in strategyproof mechanisms (i.e. mechanisms in which no agent can gain by reporting false labels). Since such mechanisms are typically suboptimal, we study the best approximation ratio that such a (deterministic) mechanism can guarantee. For example, it is easy to see that labeling the entire dataset according to the opinion of a single arbitrary agent (a dictator) is SP, but does not guarantee any finite approximation ratio. We next show that this holds for any SP mechanism.

Reducing Linear classification to Facility location. Suppose that the dataset contains \( k \) samples in generic state on the real line \( \mathbb{R}^1 \). There are exactly \( 2^k \) ways to classify the points, as all negative labels must be on one side of the classifier and likewise for the positive labels. Requiring realizability simply means that the opinion of each agent on the correct classification can be described by one of these \( 2^k \) classifiers.

The cycle \( R_{2k} \) (embedded in \( C_k \)) contains the \( 2k \) vectors \( 0^{k_1}1^{k_2} \) and \( 1^{k_1}0^{k_2} \), where \( k_1 + k_2 = k \). Note that these are exactly all the possibilities to classify the dataset with a 1-dimensional linear classifier. We can map the classification problem to a facility location problem by mapping each data point to a particular dimension of the cube \( C_k \). The opinion of each expert can be then naturally mapped to a vertex of \( C_k \). Moreover, due to realizability, this vertex lies on \( R_{2k} \). We get that any SP mechanism for classification is in fact an SP facility location mechanism on the cycle \( R_{2k} \) (the facility is the vertex representing the selected classifier). The following corollary then follows from our main theorem (through Corollary 5.2).

**Corollary 6.8.** Every SP classification mechanism for linear classifiers in \( \mathbb{R}^d \) (for any \( d \geq 1 \)) has an approximation ratio of \( \Omega(n) \), even when datasets are individually realizable.

This result is stronger than the results of Meir et al. [2012, 2010], which only holds if non-realizable datasets are allowed. For a more detailed discussion on SP linear classifiers (including a simple reduction from \( \mathbb{R}^d \) to \( \mathbb{R}^1 \)), see [Meir et al. 2010].

7. DISCUSSION

Our two primary results are the complete characterization of onto SP mechanisms on the discrete line, and proving that on sufficiently large cycles, every onto SP mechanism must be close to a dictatorship. We believe that the outline of our proofs can be easily followed and perhaps assist in the construction of similar characterization results in other domains.

For cycles, we further studied how the dictatorial limitation is affected by the cycle size and the number of agents (see Table I), specifying exact size of the cycle required to
allow mechanisms that are non-1-dictatorial. Interestingly, this effect of the cycle size is not completely monotone, presumably due to the additional effect of parity. Finally, we completed the characterization of onto SP mechanisms for even-sized cycles in these cases where the dictatorial condition holds.

**Future directions.** We conjecture that the characterization of line mechanisms can be extended to trees, similarly to the result from Schummer and Vohra [2004]. Note that the properties used for the characterization should be suitably extended first. Other directions include the characterization of SP mechanisms (both deterministic and randomized) and the study of their approximation bounds for a variety of topologies and optimization criteria. For example, one can think of extensions of this work to weighted line and cycle graphs (or weighted graphs in general), e.g. the case in which the locations on the line are 1, 2, 4, ..., 2^k. Our methods in this work (as well as in the work that preceded us for continuous graphs) relied on a certain ‘symmetry’ in the underlying distances. Thus studying such graphs with non-uniform distances may yield a better understanding of SP mechanisms.

An intriguing open question is whether randomized SP mechanisms (on a particular structure) must also be close to a random dictatorship, as we already know to be true for other domains [Gibbard 1977; Meir et al. 2011].

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A. APPENDIX FOR SECTION 3

Lemma 3.5. Every SP mechanism is monotone.

Proof. Suppose that \( f \) is SP, and assume toward a contradiction that \( f \) is not monotone. Thus there are \( a \in V^n, j \in N, b_j > a_j \) such that \( x' = f(a_{-j}, b_j) < f(a) = x \). We have that \( |x' - a_j| \geq |x - a_j| \), since otherwise \( j \) can benefit by reporting \( b_j \) in the profile \( a \). Since \( x' < x \), this implies \( x' < a_j < b_j \). Thus \( x - b_j < |x' - b_j| \), and thus \( j \) can benefit by reporting \( a_j \) instead of \( b_j \). \(\square\)

Lemma 3.6. A monotone mechanism \( f \) is Pareto iff it is unanimous.

Proof. Pareto clearly entails unanimity. It remains to show that unanimity implies Pareto. Note that the notion of Pareto on a line is equivalent to \( f(a) \in [\min_{j \in N} a_j, \max_{j \in N} a_j] \). Let \( a \in L^n, a' = \min_{j \in N} a_j \) and \( a'' = \max_{j \in N} a_j \). Also let \( a' = (a', \ldots, a') \) and \( a'' = (a'', \ldots, a'') \). By unanimity, \( f(a') = a' \) and \( f(a'') = a'' \). By monotonicity,

\[
\min_{j \in N} a_j = a' = f(a') \leq f(a) \leq f(a'') = a'' = \max_{j \in N} a_j,
\]

as required. \(\square\)

Lemma 3.7. Every SP, unanimous mechanism for the line is 1-SI.

Proof. Since \( f \) is SP and unanimous, it is MON (by Lemma 3.5) and thus also Pareto (by Lemma 3.6). Suppose by way of contradiction that \( f \) is not 1-SI. Then, there exists at least one pair of profiles that violates the 1-SI property. Assume w.l.o.g. that the two profiles differ only in agent 1’s report, and also that \( a'_1 = a_1 + 1 \). By the violation of the 1-SI property, it holds that \( f(a_1, a_{-1}) = x \neq x' = f(a'_1, a_{-1}) \), while \( d([a, a'], f(a')) > 1 \). Among these pairs, let \( a, a' \) be the two profiles that maximize \( \sum_{j \in N} a_j \).

Since \( a'_1 > a_1 \), MON implies that \( x' > x \). Let \( I_a = [a_1, a'_1] \) and \( I_x = [x, x'] \). We distinguish between the following cases:

**case a:** The intervals \( I_a, I_x \) intersect on at most one point. It follows that either \( a_1 \geq x' \) or \( a'_1 \leq x \). In the former case, agent 1 can benefit by reporting \( a'_1 \) instead of \( a_1 \) in \( a \), and similarly, in the latter case, agent 1 can benefit by reporting \( a_1 \) instead of \( a'_1 \) in \( a' \). Thus, a contradiction is reached.

**case b:** One of the intervals \( I_a, I_x \) strictly contains the other. Since \( a'_1 = a_1 + 1 \) and \( f(a_1, a_{-1}) \neq f(a'_1, a_{-1}) \), the inclusion must be \( I_a \subseteq I_x \). Further, since \( d([a_1, x'], f(a')) > 2 \), it must hold that \( x \leq a'_1 - 2 \) (i.e. that \( x < a_1 \)), as otherwise \( a'_1 \rightarrow a_1 \) is a manipulation for agent 1.

Since \( f \) is Pareto, and \( x < a_1 \), then there must be some other agent (w.l.o.g. agent 2) s.t. \( a_2 \leq x \).

We define two new profiles, \( b, b' \), that differ from \( a, a' \) only by relocating agent 2 so that \( b_2 = b'_2 = x + 1 \). Denote the new output locations by \( y = f(b) \) and \( y' = f(b') \). One can easily verify that \( b_2 > a_2 \) (and clearly \( b'_2 > a'_2 \)). It, therefore, follows from MON that \( y \geq x, y' \geq x' \), and also \( y' \geq y \). We further distinguish between two sub-cases.

If \( y' > y \), then \( y' > x' \geq a'_1 + 2 \). Then \( b, b' \) is still a violating pair. However, since \( b_2 > a_2 \) and all other agents in \( a, b \) are the same, \( \sum_{i \in N} b_i > \sum_{i \in N} a_i \) in contradiction to maximality of the pair \( a, a' \).

If \( y = y' \), then consider the pair of profiles \( a, b \) and their outcomes \( x = f(a), y = f(b) \). Note that \( y = y' \geq x' > a_1 \geq x + 1 = b_2 \), and thus \( d(y, b_2) > 1 = d(x, b_2) \). Therefore \( b_2 \rightarrow a_2 \) is a manipulation for agent 2 under profile \( b \), which is a contradiction to SP. \(\square\)

---

5This is w.l.o.g a violation of part (b) of Definition 3.2. The other, symmetric, case is when \( d([a, a'], f(a)) > 1 \).
Lemma A.1. Every 0-SI, MON mechanism $f$ for the line is SP.

Proof. Let $a$ be a profile, $j$ an agent, and $a'_j$ a deviation of $j$ and assume that $f(a_j, a_{-j}) \neq f(a'_j, a_{-j})$. W.l.o.g., assume $a_j < a'_j$. From monotoncity we get that $f(a_j, a_{-j}) < f(a'_j, a_{-j})$ and from $f$ being 0-SI $a_j \leq f(a_j, a_{-j}) < f(a'_j, a_{-j}) \leq a'_j$. Therefore, we get that $j$ prefers $f(a_j, a_{-j})$ in a and hence this is not a manipulation. □

B. APPENDIX FOR SECTION 4

Definition B.1. Let $a_1, \ldots, a_n$, s.t. the minimal arc (consecutive part of the cycle) that includes all the points is of size $< k/2$. A point $x \in R_k$ is cycle-Pareto (w.r.t. the profile $a_i$), in either of the following cases:

— $x$ lies on the arc.
— $k$ is odd, the arc size is $[k/2]$, and there is an agent $i$ next to $x$, i.e., $d(a_i, x) = 1$.

If there is no such arc, every point $x \in R_k$ is cycle-Pareto.

A mechanism $f$ is cycle-Pareto, if for any profile $x = f(a_1, a_2, \ldots, a_n)$ is a cycle-Pareto outcome.

Lemma B.2. For an odd cycle, a profile $(a_1, \ldots, a_n)$, and a point $x$:

i If $x$ is a cycle-Pareto outcome, then there is no point $y$ s.t. $d(a_i, y) < d(a_i, x)$ for every agent $i$.

ii If $x$ is not a cycle-Pareto outcome, then there exists a point $y$ s.t. $d(a_i, y) < d(a_i, x)$ for every agent $i$ and $d(a_i, y) < d(a_i, x)$ for some agent $i$.

Proof.

i Clearly, if $x$ is a cycle-Pareto outcome due to the first condition (lies on the arc), it is also Pareto.

Otherwise, $x$ does not lie on the arc, the arc size is exactly $[k/2]$, and there is an agent $s.t. d(a_i, x) = 1$. W.l.o.g, the agents are ordered clockwise $a_1, a_2, \ldots, a_n$ s.t. $d(a_1, a_n) = \frac{k-1}{2}$, $d(x, a_1) = 1$, $d(x, a_n) = \frac{k-1}{2}$. The only point that is closer to $a_1$ than $x$ is $a_1$ itself and it is not closer to $a_n$ than $x$.

ii If $x$ is not cycle-Pareto then the minimal arc is of size $\leq (k-1)/2$ and $x$ does not lie on this arc. W.l.o.g, the agents are ordered clockwise $a_1, a_2, \ldots, a_n$ and $a_1$ is the closest to $x$. I.e., $d(a_1, x) \leq d(a_i, x)$ for all $i$. Denote by $t = d(x, a_1) > 0$.

If $t + d(a_1, a_n) \leq (k-1)/2$ (and all the points lie on a semi-cycle), then for all agents $d(a_1, a_2) \leq d(x, a_1)$ and $d(a_1, a_1) = 0 < d(x, a_1)$ so $y = a_1$ satisfies the conditions.

If $t > d(a_1, a_n)$, then for all agents $d(a_1, a_n) \leq d(a, a_n) < d(a_1, x)$ so $d(x, a_1) = \min (d(a_1, a_1) + d(a_1, x), d(a_1, a_n) + d(a_1, x)) \geq \min (d(a_1, x), d(a_1, a_n)) = d(a_1, a_n)$ and $d(a_1, a_n) = 0 < d(x, a_n)$ so $y = a_n$ satisfies the conditions.

Otherwise, the point $y$ is defined as the point on the arc $[a_1, a_n]$ s.t. $d(a_1, y) = t - 1$. For any agent: If $d(a_1, a_1) + d(a_1, x) < k/2$ then $d(a_i, y) < d(a_i, x)$. Otherwise, $d(a_1, y) - d(a_1, x) = (d(a_1, a_1) - d(a_1, x) + 1) - (k - d(a_1, a_n) - d(a_1, x)) = 2d(a_1, a_1) - k + 1 \leq 0$ so $y$ satisfies the conditions.

□

\(^2\)Notice that the minimal arc is uniquely defined in such case.

\(^3\)In the literature this criterion is usually referred to as ‘Strong Pareto Dominance’ and is equivalent to definition 2.3.

\(^4\)In the literature this criterion is usually referred to as ‘Weak Pareto Dominance’.
**Lemma B.3.** If \( f \) is an SP mechanism for \( R_k \) for \( n > 2 \) agents that does not satisfy cycle-Pareto then there exists an SP mechanism \( g \) for \( R_k \) for 2 agents that does not satisfy cycle-Pareto.

**Proof.** Let \((a_1, \ldots, a_n)\) be a profile s.t. \( x = f(a_1, \ldots, a_n) \) is not a cycle-Pareto outcome. So we know that all the points lie on an arc smaller than \( k/2 \). W.l.o.g. assume \( a_1, a_2 \) are the extreme points of this arc. We define \( g \) by \( g(u, v) = f(u, v, a_3, \ldots, a_n) \).

Since \( f \) is SP, so is \( g \) and clearly \( x = g(a_1, a_2) \) is not a cycle-Pareto outcome. \( \square \)

**B.1. Two agents**

**Lemma B.4.** Let \( f \) be onto and SP rule on \( R_k \) \((k \geq 13)\). Suppose that \( x = f(a, b) \) is violating cycle-Pareto. Then \( x \) is at distance (exactly) 2 from some agent, and agents are almost antipodal, i.e. \( k/2 > d(a, b) \geq k/2 - 1 \).

**Proof.** As our proof will show, this will be true for any number of agents. However we consider the case of two agents first. Let \( f(a, b) = x \) such that \( x \) is not cycle-Pareto. Moreover, let \( a, b \) be the profile minimizing \( d(a, b) \) under this condition. W.l.o.g. \([a, b]\) is the shorter arc (we denote \( a < b \), thus \( x \in (b, a) \)). By unanimity, \( d(a, b) \geq 5 \), as otherwise there is a manipulation \( a \to b \) or vice versa (as either \( d(a, x) > 4 \) or \( d(b, x) > 4 \)). We denote \( u = f(a + 1, b) \) and \( w = f(a, b - 1) \). By minimality of \( d(a, b) \), \( u, w \) are cycle-Pareto (w.r.t. their respective profiles). See Figure 3 for an illustration.

We prove the following series of claims.

- \( u \neq w \) (W.l.o.g. that \( u = b \))
  - Indeed, suppose that they are equal, then \( u = w \in [a, b] \), and thus \( d(a, u) + d(b, u) = d(a, b) \). Also, from SP we have that \( d(a, x) \leq d(a, u) \) and \( d(b, x) \leq d(b, u) \). By joining the inequalities,
    \[
    d(a, b) = d(a, u) + d(b, u) \geq d(a, x) + d(b, x) = k - d(a, b).
    \]
    This entails that \([a, b]\) is the long arc, which is a contradiction.
  - Either \( w = a \) or \( u = b \).
    - Consider \( q = f(a + 1, b - 1) \). If \( u \neq b \), then \( q = f(a + 1, b - 1) = f(a + 1, b) = u \) by Lemma 4.1. Similarly, if \( w \neq a \), then \( q = f(a + 1, b - 1) = f(a, b - 1) = w \). Therefore if neither of the two equalities holds then \( u = q = w \) in contradiction to the previous claim.
  - \( d(q, b - 1) \leq 1 \)
    - Otherwise \( b - 1 \to b \) is a manipulation for agent 2 (under \( a + 1 \)).
- \( d(q, b) = d(q, b - 1) + 1 \leq 2 \).
- \( a \neq w \)
  - If \( a = w \), then \( d(q, a + 1) \leq 1 \), i.e. \( q = a + 1 \) or \( q = a + 2 \), and thus (since \( k \geq 13 \)) \( d(q, b) = d(a, b) - d(q, a) \geq 5 - 2 = 3 \), in contradiction to the previous result.
  - \( 1 \leq d(x, b) \leq 2 \)
    - Deviation of agent 1 \( a \to a + 1 \) is not beneficial (under \( b \)) and hence \( d(x, b) \leq d(w, b) = d(q, b) \leq 2 \)
  - \( d(x, b) = 2 \)
    - If \( k \) is even, then \( d(a, b) < k \), as otherwise every outcome is Pareto. It then follows by Lemma 4.3 that \( d(x, b) > 1 \) (i.e. \( d(x, b) = 2 \)). Similarly, if \( k \) is odd, then \( d(a, b) \leq \lfloor k/2 \rfloor \) (otherwise \( a \to b \) is a manipulation). Then \( d(b, x) = 2 \) as well, since \( d(a, x) = 1 \) would not violate cycle-Pareto by definition. Thus we get the first part of the lemma.
  - \( d(a, b) \geq k/2 - 1 \). Suppose otherwise, i.e. that \( d(a, b) < k/2 - 1 \). Since \( d(x, b) = 2 \), then \( d(a, x) > d(a, b) \), and \( a \to b \) is a manipulation for agent 1.
Fig. 3. An illustration of the original violating profile, where \( x = f(a, b) \). Agents’ locations \((a, b, \text{etc.})\) appear outside the cycle, and facility locations \((x, x', \text{etc.})\) appear inside. The conclusion of this part of the proof is that the facility must be close to one of the agents (appears in brackets).

Fig. 4. Illustrations of profiles defined in the second part of the proof.

Finally, due to Lemma B.3 SP mechanisms on \( R_k, k \geq 13 \) must be cycle-Pareto, for every \( n \).

Lemma 4.4 Let \( k \geq 13, n = 2 \). If \( f \) is SP and onto on \( R_k \), then \( f \) is cycle-Pareto.

Proof of Lemma 4.4 for \( k \geq 13 \). Recall that by Lemma B.4 cycle-Pareto can only be violated when the facility is at distance (exactly) 2 from some agent, and agents are almost antipodal.
Finally, it follows that agent 2 has a manipulation by moving from 

\( a' \) = \( x + 3 = b + 5 \) and \( a'' = x + 5 = b + 7 \) (since \( k \geq 13 \) it holds that \( a', a'' \in [x, a] \)). 

We claim that the existence of profiles \( (a', b) \) and \( (a'', b') \) leads to a contradiction. This claim completes our proof, and will also be used in the proof of Lemma 4.5.

For each of the profiles \( a', b \) and \( a'', b' \), we move agent 2 counterclockwise (away from \( x \)), and denote by \( b' \) [respectively, \( b'' \)] the first step s.t. \( f(a', b') \neq x \) [resp., \( f(a'', b'') \neq x \)]. See Figure 4(a) for an illustration.

Indeed, the facility cannot move before agent 2 crosses the point antipodal to \( a'' \) (i.e. while \( ||b - t, a''|| < ||a'', b - t|| \)), as otherwise we would have a profile violating cycle-Pareto, where \( d(x, a''), d(x, b - t) \neq 2 \). Similarly, the facility must move after the crossing (i.e. when \( ||b - t, a''|| > ||a'', b - t|| \)). Thus for odd \( k \) we get that \( b'' = a'' - \lceil k/2 \rceil \). For even \( k \), the location of the facility can be anywhere when \( a'', b'' \) are exactly antipodal, thus \( b'' \in [a'' - k/2 - 1, a'' - k/2] \).

We can summarize both cases with the following constraint:

\[
d(b'', x) = ||b'', x|| = ||b'', a''|| - 5 = \left( \frac{k}{2} + \frac{1}{2} \right) - 5 \leq \frac{k}{2} - 4.
\] (1)

A similar analysis for \( a', b' \) shows the following:

\[
d(b', x) = ||b', x|| = ||b', a'|| - 3 = \left( \frac{k}{2} + \frac{1}{2} \right) - 3 \geq \frac{k}{2} - 3.
\] (2)

— We denote \( x' = f(a', b'), x'' = f(a'', b''). \) By Lemma 4.2 \( x' \) is roughly the same distance from \( b' \) as \( x \) is, i.e.

\[
d(b'', x'') = d(b'' - 1, x) \pm 1 \leq d(b''', x) \leq k/2 - 4 \quad \text{(By Eq. (1))}
\]

\[
d(b', x') = d(b' - 1, x) \pm 1 \geq d(b', x) - 2 \geq k/2 - 5 \quad \text{(By Eq. (2))}
\]

We get that either (i) \( x'' \) is strictly closer that \( x' \) to \( b' \) (i.e. \( x'' < x' \)), or (ii) \( x'' = x', b'' = b' + 1 \). Also note that by Lemma 4.2 \( d(b', x') \leq d(b' - 1, x) \), which entails (for \( k \geq 13 \)) that \( x' \geq a'' \). See Figure 4(a) for an illustration.

(i) Suppose that \( x'' > x' \). This case leads to a simple contradiction. Denote \( z = f(a'', b') \). By Lemma 4.1, \( z = f(a'', b'') = x'' \). Similarly, since \( x' \geq a' \), \( z = f(a', b') = x' \), in contradiction to the assumption.

(ii) Suppose that \( x' = x'', b' = b' - 1 \) (see Figure 4(b)). In particular it must hold that \( k \) is even and \( d(a', b') = k/2 \), i.e. \( k \geq 14 \). We move agent 1 two steps from \( x' \) counter clockwise, to \( x + 1 \). We denote this location of agent 1 by \( a^* \). Since \( b'' > b' \), \( f(a', b') = x \) and thus by Lemma 4.1 \( f(a^*, b'') = x \) as well.

We argue that \( y = f(a^*, b') \neq x' \). Suppose otherwise, then \( y = x' \) is violating cycle-Pareto. Moreover, \( d(a^*, b') < k/2 - 1 \leq d(a, b) \) in contradiction to minimality. It thus follows that \( y \in [b', a^*] \). We further argue that \( y \in [b', x) \). Indeed, if \( y \in [x, a^*] \), then \( d(a^*, y) \leq 3 \). On the other hand,

\[
d(a^*, x') = d(a^*, b') - d(b', x') = k/2 - (k/2 - 5) = 5,
\]

thus \( a^* \rightarrow a^* \) is a manipulation for agent 1 under \( b' \).

— Finally, it follows that agent 2 has a manipulation by moving from \( b'' \) to \( b' \) under \( a^* \).

This is since the facility moves from \( x = f(a^*, b'') \) (where \( d(x, b'') \geq 2 \)) to \( y = f(a^*, b') \), where \( y \in (x, b'] \) and thus \( d(y, b'') < d(x, b'') \).
Fig. 5. An illustration of the profiles on $R_{24}$. Note that $c_{b'}$ must be roughly half way between $b'$ and $a$ (along the long arc), and likewise for $c_{b''}$ w.r.t. $[b', a]$. Moving along the long arc, agent 2 must change the critical point $c_b$. Thus there is at least one step along this path that is a manipulation for agent 2.

B.2. Three agents

**Theorem 4.9** Assume $k \geq 22$, $n = 3$. Let $f$ be an onto SP mechanism on $R_k$, then $f$ is a 1-dictator.

**Proof** for $k \geq 22$. Assume, toward a contradiction, that there is no 1-dictator.

We begin from a profile where $a = 0$, $b'' = 5$. Let $x = f(a, b'', a)$. By Lemma 4.7, $d(x, a) \leq 1$. Moreover, we can assume w.l.o.g. that $x = a = 0$, since otherwise we can move all agents toward $x$.

We also define an alternative profile, where $b' = b'' + 3 = 8$. See Figure 5 for an illustration.

We now move agent 3 counterclockwise from $a$ to $b'$ (i.e., along the long arc). Suppose $c_0 = a - 3$. Then the facility is either near $a$ or near $c_0$. If it is near $c_0$, then it must follow agent 3 all the way to $b'$ (otherwise he will have a manipulation). In particular, the facility must visit one of the points $b' + 2, b' + 3, b' + 4$. However since $k \geq 22$, then all three points are at distance of at least 10 from $a$. This means that agent 1 can manipulate $a \rightarrow b'$, bringing the facility (by Lemma 4.7) to $b' \pm 1$.

Therefore, $f(a, b', c)$ is either near $a$ or near $b'$ for all $c \in [b', a]$. Moreover, when it is near $a$ we can still assume it equals $x = a$ (since otherwise we again move agents 1 and 2 toward the facility).

We can now apply exactly the same argument on the initial profile $(a, b'', c_0)$, thus $f(a, b'', c)$ is either near $a$ or near $b''$ for all $c \in [b'', a]$.

Next, let $c_b'$ denote the last point on the long arc s.t. $f(a, b', c_{b'}) = x$, meaning that $d(x', b') \leq 1$, where $x' = f(a, b', c)$ for all $c \in (b', c_{b'})$. Similarly, $c_{b''}$ will denote the critical switching point when agent 2 is at $b''$, and $x'' = b'' \pm 1$ is the new location of the facility after the switch.

Our plan is as follows: First to show that $c_{b'}, c_{b''}$ are roughly on the middle of the long arc $[b', a]$ and $[b'', a]$, respectively. It will then follow that $c_{b'} > c_{b''}$, which will lead to a manipulation of agent 2.

Let us study the constraints on $c_b$ (for any location $b$ of agent 2). For convenience, we denote $r_b = k - c_b = d(c_b, a)$. The location of the facility after the switch is denoted by $z$. Indeed, $r_b = d(x, c_b) \leq d(z, c_b)$, otherwise there is a manipulation $c_b \rightarrow c_b + 1$ for agent 3. Thus,

$$r_b = d(x, c_b) \leq d(z, c_b) = d(b, c_b) + d(b, z) \leq d(b, c_b) + 1 = k - b - r_b + 1,$$
Similarly, \( d(c_b + 1, z) \leq d(c_b + 1, x) = r_b + 1 \) (otherwise \( c_b + 1 \rightarrow c_b \) is a manipulation), thus
\[
r_b \geq d(c_b + 1, z) - 1 = d(c_b, z) - 2 \geq k - b - r_b - 1 - 2 = k - b - r_b - 3,
\]
i.e., \( r_b \geq \frac{k - b - 3}{2} \).

Thus either \( c_r \), i.e., Theorem 6.7.

C. APPENDIX FOR SECTION 6

B.3. Small cycles

Proposition 4.11. There is an anonymous SP mechanism for two agents on \( R_k \), for all \( k \leq 12 \). For \( k \leq 9 \), we can use the same “median-like” mechanism described next, adding a third dummy agent at an arbitrary location. Tie-breaking is made in favor of the dummy agent, and then clockwise. For \( k \in \{10, 11, 12\} \) we provide a tabular description of the mechanisms online [A].

Proposition 4.12. There is an anonymous SP mechanism for three agents on \( R_k \), for all \( k \leq 14 \) and \( k = 16 \).

Proof for \( k \leq 7 \). We define a “median-like” mechanism as follows: let \( (a_3, a_1) \) be the longest clockwise arc between agents, then \( f(a) = a_2 \). Break ties clockwise, if needed. An agent (say \( a_1 \)) can try to manipulate. The current dictator (say \( a_2 \)) must be the one closer \( a_1 \), as otherwise \( ||(a_1, a_2)|| > ||(a_3, a_1)|| \). Thus changing the identity of the dictator from 2 to 3 cannot benefit agent 1. The only way to gain is to become the dictator, by moving away from \( a_2 \), making the arc \( [a_3, a_1] \) smaller. For agent 1 to become the dictator, the size of \( (a_2, a_3) \) must be at least 3, otherwise there is a longer arc (as the sum is \( k = 7 \)). We have that \( ||(a_3, a_1)|| \geq ||(a_2, a_3)|| \geq 3 \), since agent 2 is the dictator for \( a \). As a result, \( ||(a_1, a_2)|| \leq 7 - 3 - 3 = 1 \), i.e. either \( a_1 = a_2 = f(a) \) (in which case clearly there is no manipulation), or \( a_1 = a_2 - 1 \). In the latter case, we will have that \( f(a') = a'_1 \neq a_1 \), so it is still not an improvement for agent 1.

It is easy to verify that the mechanism also works for smaller cycles. \( \square \)

For \( k \in \{8 - 14, 16\} \) we provide mechanisms as three-dimensional arrays in Matlab format [A].

C. APPENDIX FOR SECTION 6

Theorem 6.7. Let \( 2k \geq 18 \) (or \( 2k \geq 12 \) for \( n = 2 \)). An onto mechanism on the cycle \( R_{2k} \) is SP if and only if it is 1-dictatorial, Cube-monotone (CMON), and IDA.
PROOF. For the first direction, we must prove that every onto SP mechanism must be IDA and CMON (in addition to being 1-dictatorial). Suppose that some agent violates CMON on coordinate $i$. This is either by crossing the location of the facility (i.e. $x \in [a_j, a_j']$), or the antipodal point (i.e. $x + k \in [a_j, a_j']$). In the first case, this is clearly a manipulation, as shown in Lemma [3.5] In the latter case, assume w.l.o.g. that $x = 0$ and $j$ moved clockwise. By Lemma [4.1] the agent moved along the longer arc $[a_j, x]$, thus $|[x, a_j]| \leq k$. By violation of CMON, the facility also moved clockwise, thus getting closer to $a_j$, which is a manipulation.

Now suppose IDA is violated by agent $j$ (moving w.l.o.g. clockwise). This means that $[a_j, a_j']$ does not contain neither $x$ nor $k + x$. If $j$ is the dictator, then this is clearly a violation (again, as in Theorem [3.4]). Otherwise, the facility can only move one or two steps. If the facility moves clockwise, then $a_j \rightarrow a_j'$ is a manipulation. Otherwise, $a_j' \rightarrow a_j$ is a manipulation.

In the other direction, we show that if $f$ is 1-dictatorial (where agent 1 is the dictator), IDA and CMON, then it must be SP. Indeed, suppose that agent $j$ moves from $a_j$ to $a_j'$, thereby moving the facility from $x$ to $x'$, where w.l.o.g. $x = 0$. If $j = 1$, then the only way to gain is by moving one step, bringing the facility to $a_1$. However this would contradict IDA. Therefore assume that $j$ is not the dictator. By Lemma [4.1] $j$ must move along the longer arc $[a_j, x]$. Consider first $a_1 = x = 0$. Note that the facility can only move to $x' = 2k - 1$ (if $x'(k) \neq x(k)$) or $x' = 1$ (if $x(1) \neq x'(1)$).

(I) If $j$ is not crossing neither $x = 0 = 0^k$, nor $k = 1^k$. Then by IDA coordinates 1 and $k$ of the facility cannot change. Since changing any other coordinate puts $x'$ at distance at least 2 from the dictator, the facility cannot move.

(II) If $a_j = x$, then $j$ cannot gain.

(III) $k \in [a_j, a_j']$ (w.l.o.g. $j$ is moving clockwise, from $a_j \in [1, k]$ to $a_j' \in [k, 2k - 1]$).

We part the movement to $a_j \rightarrow k$, and $k \rightarrow a_j'$, and denote $f(a_{-j}, k) = y$. In the first part, only coordinates between 2 and $k$ of $a_j$ change (from 0 to 1). Then by IDA $y(1) = x(1) = 0$, and the facility can only move to $y = 2k - 1$. However this means that $d(y, a_j) > d(x, a_j)$, i.e. that $j$ does not gain.

Now, if $y = 2k - 1$, then by IDA $x' = 2k - 1$ as well. Therefore suppose that $y = x = 0$. Since only coordinates $< k$ change between $k$ and $a_j'$, $x'(k) = y(k)$. On the other hand, $a_j'(1) = 0 < k(1)$, thus by CMON $x'(1) \leq x(1) = 0$ as well. Therefore, $x' = x$.

It is left to handle the case where $a_1 \neq x$. Since agent 1 is a 1-dictator $d(a_1, x) = 1$, and $d(a_1, x') \leq 1$. The only difference is that in case (III) it is possible that $y = 2k - 2$ (if $a_1 = 2k - 1$). However this is still not a manipulation, as $d(y, a_j) \geq d(x, a_j)$ (rather than strictly larger). □