Essential Alternatives and Set-Dependent Preferences - An Axiomatic Approach

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October 2002

Abstract

In this paper we deal with a broad framework that covers positive and negative aspects of freedom of choice or other factors such as complexity and diversity. In fact, it allows any conceivable interpretation of set dependence. We first provide in this general setting a sufficient condition for the preference of a set $A$ over another set $B$. This condition generalizes the sufficient condition that appears in Puppe (1996, Proposition 1). We then clarify why our axioms cannot provide a characterization of a unique ordering.

Keywords: Set-Dependence, Freedom of Choice, Complexity, Axiomatization.
JEL Classification Number: D71

* We are indebted to two anonymous referees and to Clemens Puppe for their very useful comments.
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1. Introduction
The literature on extended individual choice had two major objectives both carried out using an axiomatic approach: (i) the extension of an ordering over a set of alternatives to some particular ordering over the set of all possible such sets (see, among others, Barbera, Barret and Pattanaik (1984), Barbera and Pattanaik (1984), Fishburn (1984), Kannai and Peleg (1984), Nitzan and Pattanaik (1984)). (ii) The characterization of the notion of ‘freedom of choice’ (see, for example, Baharad and Nitzan (2000), Gravel (1994), Pattanaik and Xu (1990), Puppe (1995, 1996), Sen (1988)). The former concern yielded axiomatizations inspired by alternative criteria determining preferences over sets. The latter concern resulted in axiomatizations that initially emphasized the importance of the cardinality of the set of alternatives (Pattanaik and Xu (1990)). Subsequent studies (Sen (1993), Puppe (1996), Pattanaik and Xu (1998)) criticized the exclusive focus on cardinality, suggesting to take into account the role of preferences in judgements about freedom of choice. In the literature alternative purely consequentialist or instrumentalist views of the value of freedom of choice as embodied in opportunity sets have been proposed under certainty or uncertainty conditions. Several studies pointed out that consideration of preferences is important in assessing the intrinsic value of freedom of choice. Pattanaik and Xu (1998), for example, suggest that the relevant preferences are the ones that a reasonable person might have. Our study covers freedom of choice and other possible sources of set essentiality and significance like diversity or freedom of choice, and set essentiality can but need not be based on a single or multiple relevant preferences.

Cardinality-based respect for freedom of choice took the form of monotonicity (M) with respect to inclusion of sets of alternatives. Another condition considered in Puppe (1996) is the requirement (F) that every non-empty set of alternatives contains at least one alternative the exclusion of which is undesirable, because it reduces the individual’s freedom of choice$^1$.

Although condition M recognizes the possible “positive” essentiality of an alternative due to its contribution to freedom of choice, it disregards its possible “negative” essentiality, e.g., an alternative may add to the complexity of the set or, in the context of the averaging approach, an

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$^1$ For alternative notions of effective freedom or flexibility, see Arlegi and Nieto (2001) and Romero-Medina (2001).
alternative may lower the average utility. In fact, almost every approach towards set-dependent preferences recognizes the existence of undesirable alternatives.

Condition F can be interpreted as minimal cardinality-based respect for freedom of choice. But it can also reflect appreciation of the contribution of the excluded alternative to some aspect of set-significance other than its cardinality. For example, the contribution of the excluded alternative to diversity, freedom of choice or to the average utility of the set-elements.

It therefore seems that an axiomatization of the notion of freedom of choice raises two questions. First, the interpretation of the attempted axiomatization is questionable, since it is unclear which notion is actually being axiomatized. Secondly, once set-dependence is based on the essentiality of certain alternatives, one needs to take into account the possibility, which is disregarded by Axiom M, that the elimination of some alternatives can be desirable as in the case of the averaging, the median or the maximin approach. The existing axiomatization of freedom of choice is therefore incomplete. The main objective of this paper is to present the general setting of set-dependent preferences, to provide a sufficient condition for set preferences and to clarify why the general setting does not characterize unique preferences.

The more general notion of set-dependent preferences captures any conceivable preferences related to freedom of choice, complexity, diversity or any other factor that can induce essentiality of alternatives. In our extended setting no assumption of preference monotonicity with respect to freedom of choice or diversity is introduced. We allow preference for more freedom of choice or diversity, but also preference for reduced diversity or less freedom of choice. Hence, in some situations the individual may be interested in expanding his feasible set of alternatives, whereas in other cases he may prefer that this set is contracted.

In the next section we first introduce our framework where the individual’s preference relation is defined over sets of alternatives. We then present the axioms capturing the various aspects of set-dependent preferences. The two main results are presented in section 3. We conclude with a brief summary.
2. Preferences Over Opportunity Sets

An individual’s preference relation over opportunity sets\(^2\) \(\succeq\) is a set of pairs \(\{(A,B)\}\) belonging to the Cartesian product \(2^X \times 2^X\), such that \(A\) and \(B\) are non-empty, \(X\) is the set of all the alternatives and \(2^X\) is the power set of \(X\), namely the set of all subsets of \(X\).

We assume that \(\succeq\) is a pre-order, that is, it is reflexive (\(\forall S \subseteq X: S \succeq S\)) and transitive (\(\forall S, T, V \subseteq X: S \succeq T\) and \(T \succeq V \Rightarrow S \succeq V\)).

2.1 Set Dependence

The fact that individual choice depends on freedom of choice was noted by several scholars (see, for example, Gravel (1994), Pattanaik and Xu (1990), Puppe (1995, 1996)). Alternative set-dependent assumptions that imply different notions of freedom of choice were proposed. According to Gravel (1994), an opportunity set \(A\) offers at least as much freedom of choice as opportunity set \(B\) whenever \(A \supseteq B\) and \(A\) offers strictly more freedom of choice than \(B\) whenever \(A \supset B\). A similar approach appears in Pattanaik and Xu (1990) who suggest that for all distinct options \(x\) and \(y\), \(\{x,y\}\) offers more freedom of choice than \(\{x\}\). More recently, Puppe (1996) characterized preference for freedom of choice using the following two axioms. The first Axiom M can be referred to as weak positive monotonicity:

\[ \text{Axiom M : For all non-empty sets } A, B: B \subseteq A \Rightarrow A \succeq B \]

The second Axiom F can be referred to as minimal positive monotonicity:

\[ \text{Axiom F : For any non-empty set } A, |A| > 1, \text{ there exists } x \in A \text{ such that } A \succ A \setminus \{x\} \]

These axioms were motivated by a concern about the appropriate definition of the intrinsic value of freedom of choice. They were interpreted to imply that increasing the range of choice can only be desirable. In our view the approach based on Axioms M and F is questionable and of restricted generality because it focuses just on one aspect of set-dependence, namely, freedom of choice, and even with respect to this aspect a narrow view is taken: freedom of choice is always considered desirable in some strict or weak sense.

\(^2\) In our context of choice under complete information the set of alternatives is referred to as opportunity set.
We do not find the assumption of (global) positive monotonicity appealing because it ignores the likely existence of aversion to diversity or complexity, which is often positively related to the size of the set of alternatives. Set expansion need not therefore be desirable\(^3\). Positive monotonicity is irreconcilable with aversion to complexity or diversity which is common when the opportunity set is sufficiently large and an individual who enjoys freedom of choice may still consider diversity or complexity as a “bad”. When complexity or diversity aversion more than counterbalances his respect for freedom of choice, positive monotonicity is not satisfied. Even if the individual’s preferences are cardinality-based, monotonicity (global positive or negative monotonicity) is not necessarily a plausible assumption when these preferences reflect appreciation of freedom of choice as well as aversion to diversity or complexity. In the presence of these contrasting effects, minimal positive monotonicity (Axiom F) can be further weakened by requiring that preferences satisfy minimal positive or minimal negative monotonicity. Such a weakening of Axiom F is especially in order when preferences over opportunity sets hinge not only on the cardinality of these sets but also on the nature of the alternatives comprising them, as in the case of most of the preference relations over sets discussed and axiomatized in the literature concerned with the extension of an ordering over a set to the power set\(^4\). The notion of set-dependence in that literature is based, for example, on the average, median, maximal, minimal, or mixed max-min utility of the alternatives in the opportunity sets.

In general, comparison of opportunity sets may take into account any relevant aspect: freedom of choice, diversity and complexity being just three possible examples, and these aspects can be represented by the cardinality of the sets and by the nature of the alternatives they contain. Note that complexity and diversity of a set may depend both on the number and the nature of the alternatives in the set.

\(^3\) The framework suggested in Neme, Nieto and Quintas (1996) captures the possible negative effect of complexity.

3. Set-Dependent Preferences: Characterization

3.1 The Fundamental Axioms

In light of the discussion on the determinants of set dependence, it is clear that adding an alternative to an opportunity set may positively or negatively affect the appreciation of the set. It is also possible that such an addition has no effect on the desirability of the set. Formally, for every non-empty set \( S \subseteq X, |S| > 1 \), let:

\[
E^+(S) = \{ x \in S : S \setminus \{x\} \succ S \}, \quad E^-(S) = \{ x \in S : S \setminus \{x\} \prec S \}, \quad E^0(S) = \{ x \in S : S \sim S \setminus \{x\} \}
\]

The set of essential alternatives of the set \( S \) is denoted \( E(S) \), where \( E(S) = E^+(S) \cup E^-(S) \). Obviously, \( S = E^+(S) \cup E^-(S) \cup E^0(S) = E(S) \cup E^0(S) \). The notion of essentiality is the basis of set-dependent preferences. This essentiality enables us to present our main Axiom \( E^* \) that requires preferences to be minimally set-dependent. Preferences that satisfy this axiom may reflect dependence on the cardinality of the opportunity sets and on the nature of the alternatives and the interdependence among them. That is, they can take into account considerations related to freedom of choice, complexity, diversity or any other ‘source’ of essentiality. Such set-dependent preferences must satisfy the requirement that the addition of some alternative positively or negatively affect the desirability of the opportunity set. That is, in each set there must exist an alternative which is “positively essential” or “negatively essential”.

**Axiom \( E^* \):** For every non-empty set \( A \), \( E(A) \neq \emptyset \), that is, there exists an alternative \( x \in A \), such that \( x \in E^+(A) \) or \( x \in E^-(A) \).

Axiom \( E^* \) is minimal because its only requirement is that the individual considers at least one alternative as valuable or as a burden, taking into account the net effect of all factors determining set-dependence. Axiom \( E^* \) does not require that preferences are globally monotonic with respect to set cardinality. In fact, the spirit of Axiom \( E^* \) is “anti-monotonic” since it requires just local or point positive or negative monotonicity. We allow preference for more freedom of choice or diversity, but also preference for less freedom of choice or complexity. In general, in some situations the individual may be interested in extending his feasible set of
alternatives, whereas in other cases he may prefer that this set is contracted and such preferences can be due to the combined effect of all set-aspects discussed in the previous section.

We assume that essentiality is expandable in the sense that it can be applied to sets as well as to single alternatives. For example, if $E^+(S)$ is the set of positively essential alternatives of the set $S$, then elimination of any alternative contained in it is undesirable. Expanded essentiality requires that the elimination of any subset of $E^+(S)$, not just the elimination of single-alternative sets, is undesirable. Formally,

**Axiom SE**: For every set $S$,

(i) $\forall T \subseteq E^+(S): S \succ S - T$,

(ii) $\forall T \subseteq E^-(S): S - T \succeq S$,

(iii) $\forall T \subseteq E^0(S): S \sim S - T$.

The following axiom (independence of inessential alternatives) assigns special significance to essential alternatives$^5$.

**Axiom I**: For every set $A$, $A \sim (E^+(A) \cup E^-(A))$

Axiom I is directly implied by Axiom SE$^6$.

Axiom I requires that set evaluation is based on the valuable alternatives as well as on the ones that are considered as a burden. Since a set $E^0(S)$ can be non-empty, it is possible that some alternatives do not affect set evaluation$^7$.

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$^5$ Our notion of essentiality and Axioms E* and I* are extensions of Puppe's notion of essentiality and his Axioms F and I.

$^6$ Let $S = E^+(S) \cup E^-(S) \cup E^0(S)$. According to Axiom SE*, $\forall T \subseteq E^0(S), S \sim S - T$. Let $T = E^0(S)$. Hence, $S \sim E^+(S) \cup E^-(S)$.

$^7$ Axiom I is satisfied by preferences which are based, for example, on the average, maximal, minimal, or mixed max-min utility of the alternatives in the opportunity set. However, preferences based on the median utility of the alternatives, do not satisfy this axiom. Suppose, for example, that the set $S$ contains five alternatives, three of them being median, one is better and one is worse than the median. These alternatives represent, respectively, $E^0(S), E^+(S), E^-(S)$. It can be readily verified that in such a case Axiom I is not satisfied.
3.2 Results

In this subsection we present a sufficient condition for set-dependent preferences, discuss its relationship with the existing axiomatization of the notion of ‘freedom of choice’, and prove why an axiomatization is impossible in the general framework.

Let $2^{X_0}$ denote the set of all non-empty subsets of $X$. Puppe (1996) defines for every preorder $\succeq$ on $2^{X_0}$ the induced domination relation $\succeq^*$ on $2^{X_0}$:

$$A \succeq^* B \iff A \succeq A \cup B$$

That is, $A$ dominates $B$ when joining $B$ to $A$ is of negative or of no value.

Our first result provides a simple sufficient condition for the preference and for the induced domination of a set $A$ over a set $B$. The preference relation $\succeq$ is set-dependent if it satisfies two minimal requirements, namely, axiom E* and axiom SE*. The sufficient condition requires that the positively essential alternatives of the union of $A$ and $B$ are contained in the preferred set $A$ and the negatively essential alternatives of the union of $A$ and $B$ belong to the inferior set $B$. Formally,

**Theorem 1:** Let $\succeq$ be a preorder on $2^{X_0}$ that satisfies Axioms E* and SE*. For all $A,B \in 2^{X_0}$, $E^+(A \cup B) \subseteq A$ and $E^-(A \cup B) \subseteq B \Rightarrow A \succeq B$ and $A \succeq^* B$.

**Proof:** Let us use the following notation:

$${E^+_A}(A \cup B)=\{x| x \in E^+(A \cup B) \text{ and } x \notin B\}$$

$${E^+_{AB}}(A \cup B)=\{x| x \in E^+(A \cup B) \text{ and } x \in A \text{ and } x \in B\}$$

Note that, by assumption, $E^+_B(A \cup B)$ is empty.

A similar notation applies with respect to $E^0(A \cup B)$ and to $E^-(A \cup B)$. For convenience, we write $E^+$, $E^-$ and $E^0$ instead of $E^+(A \cup B)$, $E^-(A \cup B)$ and $E^0(A \cup B)$.

Let $\succeq$ satisfy Axioms E* and SE*. Suppose that $E^+ \subseteq A$ and $E^- \subseteq B$. Then,

$$E^+_A \cup E^+_A \cup E^-_B \cup E^-_A \cup E^0_A \cup E^0_B \cup E^0_A \cup E^0_B = A \cup B.$$ 

We first prove that $E^+(A \cup B) \subseteq A$ and $E^-(A \cup B) \subseteq B \Rightarrow A \succeq^* B$.

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8 Puppe (1996) uses the notation $\mathcal{E}$ for the set of all non-empty subsets. In our context, $\mathcal{E} \cup \emptyset$ and $\mathcal{E}$ corresponds to $2^X$ and $2^{X_0}$, respectively. Our definition of induced preference relation is similar to Puppe’s.
According to SE*(iii), applied to P1.1 with respect to $E^0_B$,

\[ E_A^+ \cup E_{AB}^+ \cup E_B^- \cup E_{AB}^- \cup E_A^0 \cup E_{AB}^0 \sim A \cup B \]

According to SE*(ii), applied to P1.2 with respect to $E^-_B$,

\[ E_A^+ \cup E_{AB}^+ \cup E_B^- \cup E_{AB}^- \cup E_A^0 \cup E_{AB}^0 = A \preceq A \cup B \]

We now prove that $E^+(A \cup B) \subseteq A$ and $E^-(A \cup B) \subseteq B \Rightarrow A \succeq B$.

By P1.1 and SE*(iii), applied to P1.1 with respect to $E^0_A$,

\[ E_A^+ \cup E_{AB}^+ \cup E_B^- \cup E_{AB}^- \cup E_A^0 \cup E_{AB}^0 \sim A \cup B \]

By SE*(i), applied to P1.3 with respect to $E^+_A$,

\[ E_{AB}^+ \cup E_B^- \cup E_{AB}^- \cup E_B^0 \cup E_{AB}^0 = B \preceq A \cup B \]

which completes the proof.

Suppose now that we disregard the possible existence of negative essentiality, that is, assume that for every set $A$, $E^-(A) = \emptyset$. In such a case axiom E* reduces to axiom F and axiom I* reduces to Puppe’s axiom I.

**Axiom I:** For all $A$, $E^+(A) \sim A$

Using axioms F, M, and I, Puppe (1996) proposes the following axiomatization:

**Proposition 1:** Let $\succeq$ be a preorder on $2^X$ which satisfies Axiom F. Then, $\succeq$ satisfies Axioms M and I if and only if the induced domination relation $\succeq^*$ is a restriction of $\succeq$ and for all $A, B \in 2^X$: $A \succeq^* B \iff E^+(A \cup B) \subseteq A$.

**Proof:** See Proposition 1 in Puppe (1996).

It can be readily verified that this proposition is not valid in the general setting where negative essentiality is not ruled out. For example, suppose that $B = \{b\}$, $b \notin A$, $b \in E^- (A \cup B)$. In such a case, $A \succeq^* B$ and $E^+(A \cup B) \subseteq A$. However, Axiom M is not satisfied since $A \succ (A \cup B)$.

It should be pointed out that in the general setting, in most of the cases of set-dependent preferences dealt with in the literature (for example, the approaches based on the averaging, the median and the maximax), $\succeq^* \iff \succeq$, which implies that $A \succeq B \iff A \succ (A \cup B)$. Suppose, for example, that $a$ and $b$ are the minimal (that is the worst) alternatives in $A$ and $B$, respectively.
According to the maximin criterion, $A \geq B$ if $a$ is considered as better than $b$. In such a case, it can be readily verified that $A \geq (A \cup B)$. The same argument applies when preferences are based on the averaging, the median and the maximax criterion.

Finally, if negative essentiality is disregarded, then Axiom SE* implies both Axiom M and Axiom I. According to SE*, for every set $S$: $\forall T \subseteq E^{+}(S), S \sim S-T$, which covers the requirement in Axiom I (but not vice versa). In the special case where $T = E^{0}(S)$ and negative essentiality is disregarded, $S-T = E^{+}(S)$. Hence, according to SE*, $S \sim S-T \Rightarrow S \sim E^{+}(S)$, which is equivalent to Axiom I.

Another requirement of Axiom SE*, is that for every set $S$: $\forall T \subseteq E^{+}(S), S \succ S-T$. When combined with the former requirement, it implies Axiom M: every set is preferred to its subsets.

By proposition 1 we get the following alternative axiomatization of set-dependent preferences:

**Proposition 2:** Let $\succeq$ be a preorder on $2^X$ which satisfies Axiom $F$. Then, $\succeq$ satisfies Axiom SE* if and only if the induced domination relation $\succeq^*$ is a restriction of $\succeq$ and for all $A,B \in 2^X$, $A \succeq^* B \iff E^{+}(A \cup B) \subseteq A$.

Note that the sufficient condition that appears in Theorem 1 generalizes the sufficient condition in Proposition 1. However, unlike Proposition 1, Theorem 1 does not provide a characterization of $\succeq$. As we show below, in the general setting (that allows negative essentiality), the two conditions E* and SE* do not characterize a unique preordering.

**Theorem 2:** Axioms E* and SE* do not characterize a unique preordering $\succeq$ on $2^X$.

**Proof:** Let $\succeq_{av}$ and $\succeq_{mn}$ denote two preorders on $2^X$ satisfying Axioms E* and SE* that are respectively determined according to the averaging and maximin criteria. These preferences are defined as follows:

Let $a_i, i=1,\ldots,|A|$ and $b_j, j=1,\ldots,|B|$, denote typical alternatives in $A$ and $B$. Let $U(a_i)$ and $U(b_j)$ denote the utility associated with such alternatives. Then:

$$A \succeq_{av} B \text{ iff } \frac{1}{|A|} \sum_{i=1}^{A} U(a_i) \geq \frac{1}{|B|} \sum_{j=1}^{B} U(b_j)$$

and
$A \succeq_{\text{mn}} B$ iff $\min \{ U(a_i) \}_{i=1,\ldots,|A|} \geq \min \{ U(b_j) \}_{j=1,\ldots,|B|}$

These two criteria do not coincide, which completes the proof. ■

4. Summary

In this paper we have presented a general framework that allows preferences based on any conceivable source of essentiality and, in turn, of set dependence, including the appreciation of diversity, complexity or freedom of choice. We derived a sufficient condition for the preference of a set $A$ over another set $B$ and clarified why this condition does not characterize a unique preordering. Our framework generalizes the one suggested in Puppe (1996). However, it is too general to enable axiomatization of all set-dependent preferences. It is natural therefore to axiomatize not a general notion of set-dependence, but specific set-dependent preferences as indeed was done in the literature on the extension of an ordering over a set of alternatives to some particular ordering over the set of all possible such sets.
References


