ADVERTISING IN A COMPETITIVE PRODUCT LINE

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In many markets, competing but differentiated firms offer a product line in order to target each product to a different type of customer. Under competition, firms, while trying to target different products to different segments, are also interested in "stealing" customers from the competitors and, in particular, customers from the most profitable segments. How should a profit-maximising firm allocate the advertising budget along a competitive product line? Developing a closed-loop time-variant Nash equilibrium strategy for a multi-player multi-product advertising game, we find the determinants of the optimal advertising budget allocation in a product line in an oligopoly growing market. Moreover, for a symmetric competition in a fixed market, we derive an analytic feedback time-variant Nash equilibrium strategy. Marketing implications regarding the firm's strategic orientation in product preference advertising spending, as well as comparison to rivals' spending, are discussed. Finally, we show that using a single-product advertising game instead of a multi-product advertising game, leads to over-advertising.

1. Introduction

In many markets, competing but differentiated firms offer a product line in order to target each product to a different type of customer. Under competition, firms, while trying to target different products to different segments, are also interested in "stealing" customers from the competitors and, in particular, customers from the most profitable segments. How should a profit-maximising firm allocate the advertising budget along a competitive product line? The present paper deals with this major question. Assuming a dynamic and competitive environment in a growing market, we consider the case where the strategic partners focus their marketing activities (such as advertising) on attracting sales from each other and from outside competition.

An appropriate framework for analysing dynamic marketing expenditure decisions in a competitive setting is that of differential games. A good number of differential game models in advertising appear in the literature, see for e.g., Eliashberg and Chatterjee (1985), Rao (1990), and Erickson (1991). The special issues of the Journal of Marketing Research (1985) on "Competition in Marketing" and of Marketing Science (1988) on "Competitive Marketing Strategy" are instants of the interest in incorporating competitive effects in modeling the
response of firms' sales to such marketing variables as advertising. Still most of the research has been restricted to open-loop solutions, which are not always appropriate.

In this paper, we focus on deterring closed-loop solutions. To model the dynamics of the competition, resulting from the firms' marketing efforts to attract sales from competitors, we chose the combat Lanchester model. Many researchers in different contexts have used this model. Kimball (1957) first applied it to a combat problem, and then Vidale and Wolfe (1957) used it as a sales-advertising response model. Others, among them, Isaacs (1965), Horsky (1977), Little (1979), Case (1979), Deal et al. (1979), Deal (1979), Sorger (1989), Sethi (1983), Erickson (1985, 1991, 1992 and 1995), Fruchter and Kalish (1997 and 1998) and Fruchter (1999) applied it to find optimal advertising strategies in a dynamic, competitive market. Erickson (1992) provides empirical evidence that a closed-loop solution fits the data better than an open-loop solution.

As a review by Erickson (1995) indicates, much of the differential game analysis of advertising competition has been limited to duopoly and fixed market situations, and very little study of more general oligopolies has been conducted. The extension to oligopolies, even in a fixed market, is especially difficult as regards closed-loop strategies, because there are more than one-state variables. In a growing market, even in duopoly, there are more than one-state variables.

Fruchter and Kalish (1997) provide a closed-loop Nash equilibrium solution for a duopoly with fixed market, extending the analytical solution of Case (1979) to the case of nonzero discount rate. Fruchter and Kalish (1998) analyse the case of oligopoly with a fixed market and a single product using multi-instruments to influence the sales dynamics of its product. Further analyses to a single product in oligopoly markets are provided in Fruchter (1999a and 1999b).

The present study extends the methodology of finding closed-loop strategies to an oligopoly growing market with multi-products using advertising (one instrument) to influence the sales dynamics of the product line. We summarise the contributions of this paper as follows:

(a) We develop a suitable differential game that extends the single product advertising analysis of Fruchter (1999b) to the problem of advertising in a competitive product line.
(b) For this game, we find Nash equilibrium closed-loop time-variant strategies.
(c) For a symmetric competition, we go further on and find an analytic feedback (subgame perfect) time-variant Nash equilibrium strategy.

Moreover, the analysis of the model and solution lead to several marketing implications.

Specifically we find that:

(1) The market potential of the competitive product line depends on advertising efforts of all players and all products.
Determinants of optimal advertising budget allocation of a product line are:
- advertising effectiveness,
- gross profit rate,
- potential sales of the specific product
and the total number of the products in the line.

The optimal advertising expenditure, allocated to a product in the line, is monotonically increasing with the potential sales of that product, and monotonically decreasing with the number of the products in the line.

If the products differ in potential sales, advertising effectiveness or gross profit rates, then imitation of the advertising budget allocated to one product to another leads to over- or under-advertising.

If the competitors differ in potential sales, advertising effectiveness or gross profit rates, then imitation of the advertising budget allocated to the same product of one competitor to another leads to over- or under-advertising.

Using a single-product advertising game instead of a multi-product advertising game leads to over-advertising.

Next we present the model.

2. Model Description

We consider an industry in a growing market where \( n \) firms compete with a set of products\(^a\) using advertising (as their major marketing instrument) to attract customers from each other as well as new customers, in their efforts to maximise their discounted profits over the planning horizon. Assuming an infinite planning horizon, the problem of oligopolist \( k \), \( k \in \{1, \ldots, n\} \), is to develop an advertising strategy that will maximise the present value of the firm's profit stream, as given by

\[
\Pi_k(u_1, \ldots, u_n) = \int_0^\infty \sum_{i=1}^m q_k s_k(t) - A_k(t))e^{-rt}dt, \quad k = 1, \ldots, n. \tag{1}
\]

In formula (1), \( s_k(t) \), \( i = 1, \ldots, m \), represents the sales rate, at time \( t \), of product \( i \) of firm \( k \), and \( m \) is the maximal number of products in the competitive set (if for example product \( i \) of competitor \( k \) is not in competition, this means, \( s_{ki} = 0 \)); \( q_k \), \( i = 1, \ldots, m \), represents the gross profit rate of product \( i \) of firm \( k \); the amount \( A_k(t) \) represents the advertising expenditure of firm \( k \), at time \( t \), for the whole product line; and \( r \) is the discount rate. For simplicity, we assume that all the firms have the same discount rate.

There is a dynamic relation between the rate of change in sales and the competitors' simultaneous advertising efforts to attract and generate sales. A well-known relation that captures the above dynamics in the form of a differential equation is the combat Lanchester model. The studies quoted above have used this model for a single product. We hereby relax this assumption.

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\(a\)The literature uses different terms in defining a competitive set of products. Lehmann and Winer (1997) distinguish between product form and product category competition depending on whether the products have the same or similar features. A competition that focuses on substitutable product categories is termed generic competition.
Let the term \( \rho_{ki} u_{ki}(t) \) denote the effectiveness of the advertising efforts of firm \( k, k \in \{1, 2, \ldots, n\} \), targeted to product \( i, i = 1, \ldots, m \), in attracting potential sales at time \( t \). The constant \( \rho_{ki} \) is related to media of advertising, and to consumers’ brand perceptions and preferences. We identify the advertising effort of competitor \( k \), allocated to product \( i, u_{ki} \), with the square root of the corresponding advertising expenditure, i.e. the total advertising expenditure of firm \( k \) will be,

\[
A_k(t) = \sum_{i=1}^{m} u_{ki}^2(t).
\]  

Let \( N_i \) be a parameter that represents the saturation level of product \( i \) or, in other words, the limit of the entire industry sales that can be generated from product \( i \) as a result of marketing activities. Let \( \varepsilon_i(t) \) be the market potential of product \( i \) at time \( t \), i.e. the difference between the saturation level of the product \( i \)'s market, \( N_i \), and product \( i \)'s total present industry sales at time \( t \), \( \sum_{k=1}^{n} s_{ki}(t) \). Then

\[
N_i = \sum_{k=1}^{n} s_{ki}(t) + \varepsilon_i(t).
\]

Using the above notation, we generalize the Lanchester model in the following way,

\[
\dot{s}_{ki}(t) = \rho_{ki} u_{ki}(t)(N_i - s_{ki}(t)) - s_{ki}(t) \sum_{j=1, j \neq k}^{n} \rho_{ji} u_{ji}(t)
\]

or

\[
\dot{s}_{ki}(t) = \rho_{ki} u_{ki}(t)N_i - s_{ki}(t) \sum_{j=1}^{n} \rho_{ji} u_{ji}(t),
\]

\[
s_{ki}(0) = s_{ki}^0, \quad k = 1, \ldots, n, \quad i = 1, \ldots, m
\]

2.1. Problem statement

Considering (1)–(4) the oligopolist \( k \)'s problem, \( k = 1, \ldots, n \), is:

\[
\max_{u_k} \int_{0}^{\infty} \sum_{i=1}^{m} [q_{ki} s_{ki}(t) - u_{ki}^2(t)] e^{-rt} dt
\]

\[
\dot{s}_{ki}(t) = \rho_{ki} u_{ki}(t)N_i - s_{ki}(t) \sum_{j=1}^{n} \rho_{ji} u_{ji}(t), \quad s_{ki}(0) = s_{ki}^0, \quad i = 1, \ldots, m
\]

For this differential game, we want to find Nash equilibrium closed-loop strategies. Such strategies, particularly depend on the current level of the sale rates. Therefore, having the ability to capture current changes in the market, they are realistic strategies. However, while the concept of closed-loop strategy is more appealing it is more difficult to compute. Case (1979) found a closed-form, closed-loop solution.
for the duopoly advertising game for a fixed and single-product market but only for a zero discount rate. He mentioned that the oligopoly advertising game is unsolvable, even for a zero discount rate. Finding a closed-loop solution in the case of an oligopoly game is especially difficult because there is more than one state variable. Here, we solve this problem for an oligopoly growing and multi-product market.

2.2. The market potential of the line

According to (4), changes in the sale rates of product i of competitor k are a result of the advertising efforts on potential sales of firm k, \( N_i - s_{ki} \), and simultaneous opposite efforts of the rivals to generate brand switching. Note that this model does not include customer retention activities.

Considering Eqs. (3) and (4), we have

\[
\dot{\varepsilon}_i(t) = -\sum_{k=1}^{n} \dot{s}_{ki}(t) = -\varepsilon_i(t) \sum_{k=1}^{n} \rho_{ki} u_{ki}(t),
\]

\[
\varepsilon_i(0) = N_i - \sum_{k=1}^{n} s_{ki}(0), \quad i = 1, \ldots, m.
\]

Solving the differential Eq. (5a), we obtain

\[
\varepsilon_i(t) = \varepsilon_i(0) \exp \left(-\sum_{k=1}^{n} \int_{0}^{t} \rho_{ki} u_{ki}(\tau) d\tau\right)
\]

Therefore (5b) demonstrates that:

Remark 2.1. The market potential of each product in the competitive line is a function of advertising efforts of all players. Furthermore, it decreases exponentially with the effect of the aggregate advertising of the competitors' efforts. Alternatively, the total industry of sales of product i of competitor k grow exponentially with the aggregate advertising effectiveness of the competitors' efforts.

Moreover, Eq. (5b) implies that

\[
\sum_{i=1}^{m} \varepsilon_i(t) = \sum_{i=1}^{m} \varepsilon_i(0) \exp \left(-\sum_{k=1}^{n} \int_{0}^{t} \rho_{ki} u_{ki}(\tau) d\tau\right)
\]

Let

\[
\bar{N} = \sum_{i=1}^{m} N_i
\]

be the saturation level of the line. Since saturation level is related to the demand, formula (6) expresses the interdependence of demand of one product with others in the line.

Considering (3), (5c) and (6), the difference between the saturation level of the line and the total present line sales at time t, \( \bar{N} - \sum_{i=1}^{m} \sum_{k=1}^{n} s_{ki}(t) \), is given by \( \sum_{i=1}^{m} \varepsilon_i(t) \). Therefore (5c) demonstrates that:
Remark 2.2. The market potential of the line in our model is a function of advertising efforts of all players and all products that compose the competitive line.

Remark 2.3. The particular case of
\[ \epsilon_i(t) = 0, \quad i = 1, \ldots, m \]  
(7)
corresponds to the case when the market of product \( i \) is saturated and mature and the total industry sales of the product is fixed.

3. Main Results

3.1. The Nash equilibrium closed-loop strategies

Extending the approach of Fruchter (1999b) to the multi-product problem, we arrive at the following result. For details see Appendix A.

**Theorem 3.1.** Consider the differential game associated with (5) and let \( \phi(t) \) satisfy the following two-point boundary value problem (TPBVP),
\[ \phi(t) = \frac{1}{2} \rho_{ki}^2 (N_i - s_{ki})^2 q_{ki} - s_{ki} - \sum_{j=1}^{n} \rho_{ij}^2 (N_i - s_{ji}) q_{ji} \Phi_{ii} e^t \]
\[ s_{ki}(0) = s_{ki}^0, \quad k = 1, \ldots, n, \quad i = 1, \ldots, m \]  
(8)
\[ \Phi_{ii}(t) = \frac{1}{2} \sum_{j=1}^{n} \rho_{ij}^2 (N_i - s_{ji}) q_{ji} \Phi_{ii}^2 (t) e^t - e^{-rt} \]  
\[ \Phi_{ii}(\infty) = 0, \quad i = 1, \ldots, m \]
Then
\[ u_{ki}^* = \frac{1}{2} \rho_{ki} q_{ki} \Phi_{ii}(t) e^t (N_i - s_{ki}), \quad k = 1, \ldots, n, \quad i = 1, \ldots, m \]  
(9)
are global Nash equilibrium closed-loop strategies of the above differential game.

**Proof.** See Appendix B. \( \square \)

**Remark 3.1.** If \( m = 1 \), the results are consistent with Fruchter (1999b).

Considering (9) and (6), we find that the factors that affect the advertising budget allocation for a product line are: (i) advertising effectiveness, (ii) gross profit rate, (iii) potential sales of the specific product, and (iv) the total number of the products in the line.

Moreover, considering (6), since the total saturation level, \( N \), is fixed, we find that \( N_i \) is monotonically decreasing with \( m \). Considering (9), we conclude with the following result regarding the optimal allocation of the advertising expenditure of each competitor in a product line, \( (u_{ki}^*)^2, k = 1, \ldots, n, i = 1, \ldots, m \).

**Proposition 3.1.** In a competitive market, the optimal advertising expenditure allocated to a product, \( (u_{ki}^*)^2, k = 1, \ldots, n, i = 1, \ldots, m \) is monotonically increasing
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with the potential sales of that product, \( N_i - s_{ki} \), and monotonically decreasing with the number of the products in the line, \( m \).

**Remark 3.2.** The time-variant coefficient of each advertising strategy \( u_k^* \) is the same for each firm. At any moment, the advertising strategy of each firm differs only by its advertising effectiveness, gross profit rate and actual sales rate.

### 3.2. The Nash equilibrium feedback strategy

We consider now a case where the market is fixed, i.e.

\[ \sum_{k=1}^{n} s_{ki} = N_i, \quad i = 1, \ldots, m, \]

and the competitive effects are symmetrically created, i.e.

\[ \rho_{ki} = \rho, \quad \varphi_{ki} = \varphi, \quad k = 1, \ldots, n, \quad i = 1, \ldots, m. \]

This is the case when competitors, having similar products, cannot be differentiated with respect to their advertising effectiveness and gross profit rates, and in addition, the market is saturated. This leads to the following result.

**Theorem 3.2.** If the competition is symmetric and the market is fixed, then the closed-loop time-variant Nash equilibrium strategies in Theorem 3.1 become feedback time-variant Nash equilibrium strategies. Furthermore, they have the following analytical functional form, where

\[ u_{ki}^* = \frac{1}{2} \varphi \varphi_{ki} \Psi_i(t)(N_i - s_{ki}), \quad k = 1, \ldots, n, \quad i = 1, \ldots, m, \tag{12} \]

where

\[ \Psi_i(t) = a_i + \frac{1}{b_i(t)} \]

and

\[ a_i = \frac{-r - \sqrt{r^2 + 2\rho^2 \varphi_i(n - 1)N_i}}{\rho^2 \varphi_i(n - 1)N_i} \quad b_i = [r^2 + 2\rho^2 \varphi_i(n - 1)N_i]^{1/2} \]

and

\[ \gamma = \frac{\rho^2 \varphi_i(n - 1)N_i/2}{r^2 + 2\rho^2 \varphi_i(n - 1)N_i}^{1/2} \]

**Proof.** See Appendix C.

**Remark 3.3.** The equilibrium strategy (12) has the property of being subgame perfect, i.e. it is optimal not only for the original game as specified by its initial conditions but also for every subgame evolving from it.
3.3. A duopoly fixed market

If \( n = 2 \) and the market of each product is fixed, then \( s_{1i} + s_{2i} = N_i \). Let

\[
s_{1i} = s_i \quad \text{and} \quad s_{2i} = N_i - s_i, \quad i = 1, \ldots, m
\]

and let

\[
\Phi_i(t)e^{rt} = \varphi(s^*)
\]

Considering the formula

\[
\Phi_i e^{rt} + r\varphi(s^*) = \varphi(s^*)s^*,
\]

and substitution in the second equation of (8) yields

\[
\frac{1}{2}[\rho_{2i}^2 q_{1i}(N_i - s^*)_2^2 - \rho_{2i}^2 q_{2i}s^*] \varphi'(s^*) \varphi(s^*)
\]

\[
= \frac{1}{2}[\rho_{2i}^2 q_{1i}(N_i - s^*)_2^2 + \rho_{2i}^2 q_{2i}s^*] \varphi'(s^*)s^* - 1 + r\varphi(s^*),
\]

\[
\lim_{t \to \infty} \varphi(s^*_i(t))e^{-rt} = 0 \quad i = 1, \ldots, m
\]

Remark 3.4. Solving (16) backward we are able to find \( \varphi(s^*(0)) \) and therefore in a duopoly fixed market, the TPBVP (8) can be transformed into an initial value problem (which is a much easier problem).

Remark 3.5. The equilibrium strategy in Theorem 3.1 now becomes

\[
\begin{align*}

u_{1i}^* &= \frac{1}{2} \rho_{1i} q_{1i} \varphi(s^*)(N_i - s_i) \quad \text{and} \quad u_{2i}^* = \frac{1}{2} \rho_{2i} q_{2i} \varphi(s^*)s_i, \\

& \quad i = 1, \ldots, m.
\end{align*}
\]

Remark 3.6. If \( m = 1 \), taking \( N_1 = 1 \), the results here are consistent with Fruchter and Kalish (1997).

4. Marketing Implications

The results obtained in Sec. 2, shed light on the general orientation of a firm's advertising strategies, as measured by the ratio of advertising budget allocation to different products of a firm. In this context, Theorem 3.1 leads to the following result.

Corollary 4.1. Consider an oligopoly growing market, and let \( i \) and \( j \neq i, i, j \in \{1, \ldots, m\} \), be two different products in the competitive product line of firm \( k \), \( k = 1, \ldots, n \). Let \( A_{ki} \) be the advertising expenditure allocated to product \( i \) and \( A_{kj} \) to product \( j \), then

\[
\frac{A_{ki}}{A_{kj}} = \left( \frac{N_i - s_{ki}}{N_j - s_{kj}} \right)^2 \left( \frac{\Phi_i(t) \rho_{ki} q_{ki}}{\Phi_j(t) \rho_{kj} q_{kj}} \right)^2 \quad \forall \ k = 1, \ldots, n, \ \forall \ i, j = 1, \ldots, m.
\]
Another useful orientation of a firm's advertising strategies can be obtained by measuring the ratio of advertising budget allocated to the same product by different competitors. Considering again Theorem 3.1, we conclude with the following.

Corollary 4.2. Consider an oligopoly growing market. Let $A_{ki}$ and $A_{ji}$ be advertising expenditures allocated to product $i$ by firms $k$ and $j$, respectively. Then:

$$\frac{A_{ki}}{A_{ji}} = \left(\frac{N_i - s_{ki}}{N_i - s_{ji}}\right)^2 \left(\frac{\rho_{ki} q_{ki}}{\rho_{ji} q_{ji}}\right)^2, \quad \forall k, \quad j = 1, \ldots, n, \quad \forall i = 1, \ldots, n.$$ 

According to Corollaries 4.1 and 4.2, we conclude with

(a) Advertising expenditures of two products into a product line of an oligopolist are equal if and only if the products have the same potential sales, advertising effectiveness and gross profit rates.

(b) For the same product, advertising expenditures of two competitors are equal if and only if the competitors have the same potential sales, advertising effectiveness and gross profit rates.

This leads to the following result:

Proposition 4.1. In a competitive product line:

1. If the products differ in potential sales, advertising effectiveness or gross profit rates, then imitation of the advertising budget allocated to one product to another leads to over- or under-advertising.

2. If the competitors differ in potential sales, advertising effectiveness or gross profit rates, then imitation of the advertising budget allocated to the same product of one competitor to another leads to over- or under-advertising.

5. The Multi-Product versus the Single-Product Model

In the following, we want to assess the advertising expenditures of an oligopolist in a growing market with a multi-product competitive line, using the model of this study over a single-product model.

If the competitor of the multi-product game is using the single-product game, then in Theorem 3.1, $m = 1$ and according to Proposition 3.1, we conclude with the following result.

Proposition 5.1. In a competitive product line, the use of a single-product advertising game instead of a multi-product advertising game leads to over-advertising.

6. Conclusions

Developing a differential game to solve the problem of advertising budget allocation among different products of a competitive product line of a firm in a growing market, we find a closed-loop time-variant Nash equilibrium advertising strategy...
for each product in the line. The value of closed-loop strategies is that they allow
a manager to adjust to changing market conditions in an optimal way. This en-
hances the ability to deal effectively with dynamic competition in the development
of advertising strategies. The resulting strategy provides us with the determinants
of the optimal advertising budget of a product line; We also show that in a duopoly
fixed market, the mathematical computations become much simpler. In a case of
symmetric competition, with fixed market, we are able to find an analytic feedback
time-variant (subgame perfect) Nash equilibrium strategy. Marketing implications
regarding the firm’s strategic orientation in product preference advertising spend-
ing, as well as comparison to rivals’ spending, are discussed. Finally, we show that
using a single-product advertising game, instead of a multi-product advertising
game, leads to over-advertising.

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Appendix A. Derivation of the Closed-Loop Strategies

Considering the differential game (5), using variational calculus, cf. Kamien and
Schwartz (1991), and using a similar relationship between the co-state and function
Φ u we obtain the TPBVP (8). Let \(u^o_{ki} \) be the open-loop strategies. Then

\[
u^o_{ki} = \frac{1}{2} \rho_{ki} q_{ki} \Phi_{ui}(t) e^{t(N_i - s^o_{ki})}, \quad k = 1, \ldots, n, \quad i = 1, \ldots, m, \quad (A.1)
\]

where \(s^o_{ki} \) and \(\Phi_{ui}(t) \) are solutions of the TPBVP (8). From (A.1), we construct the
closed-loop strategies defined in (9).

Appendix B. Proof of Theorem 3.1

To prove this theorem we use vectorial and matrix notation. The vectors \(q_k \) and
\(s_k \) are of length \(m \) and consist from the elements \(q_{k1}, \ldots, q_{km} \) and \(s_{k1}, \ldots, s_{km} \),
respectively. The matrix function \(\Phi(t)\) is diagonal and consists of the elements
\(\Phi_{ui}(t), \quad i = 1, \ldots, m \). Consider the zero sum

\[
0 = -q_k^T \Phi(t) s_k + \int_0^\infty \frac{d}{dt} (q_k^T \Phi(t) s_k) dt .
\]

From (B.1), we obtain

\[
0 = q_k^T \Phi(0) s_k(0) + \int_0^\infty (q_k^T \dot{\Phi}(t) s_k + q_k^T \Phi(t) s_k(t)) dt
\]

Using Eq. (4) we obtain

\[
0 = q_k^T \Phi(0) s_k(0) + \int_0^\infty (q_k^T \dot{\Phi}(t) s_k + q_k^T \Phi(t)(B_k N u_k(t) - \sum_{j=1}^n B_j U_j(t) s_k) dt
\]
In (B.3), $B_k, U_j$ and $N$ are diagonal matrices consisting of $\rho_{k1}, \ldots, \rho_{km}, u_{jl}, \ldots, u_{jm}$, and $N_1, \ldots, N_m$, respectively. Considering (8) for $\Phi$, we obtain

$$0 = q_k^T \Phi(0)s_k(0) + \int_0^\infty q_k^T \left( \sum_{j=1}^n B_j^2 Q_j(N - \Omega_j^2(t))\Phi^2(t)e^{rt} - I e^{-rt} \right) s_k$$

$$+ q_k^T \Phi(t) \left( B_k N u_k(t) - \sum_{j=1}^n B_j U_j(t)s_k \right) dt$$

where $\Omega_j^2$ is a diagonal matrix consisting of the element $s_{j1}^2, \ldots, s_{jm}^2$.

Considering (B.1) and (9), Eq. (B.4) becomes

$$0 = q_k^T \Phi(0)s_k(0) + \int_0^\infty \left[ 2(u_k - u_k^o)^T u_k^* + q_k^T B_k \Phi(t) Ne^{rt} u_k^* o_k^T - q_k^T \Phi(t)e^{-rt} \left( \sum_{j=1}^n B_j (u_j - u_j^o) + 2 \sum_{j=1}^n (u_j - u_j^o)^T B_j B_k^{-1} u_k^* - q_k^T s_k \right) \right]$$

$$\times e^{-rt} dt$$

Adding the zero sum (B.5) to $\Pi_k = \Pi_k(u_1, \ldots, u_n)$, in (1), we obtain

$$\Pi_k(u_1, \ldots, u_n) = q_k^T \Phi(0)s_k(0) + \int_0^\infty \left[ -||u_k - u_k^o||^2 + ||u_k^o||^2 - 2u_k^o u_k^* \right. \left. + q_k^T B_k \Phi(t) Ne^{rt} u_k^* o_k^T - q_k^T \Phi(t)e^{-rt} \left( \sum_{j=1}^n B_j (u_j - u_j^o) + 2 \sum_{j=1}^n (u_j - u_j^o)^T B_j B_k^{-1} u_k^* - q_k^T s_k \right) \right]$$

$$\times e^{-rt} dt$$

where $\|o\|^2$ represents the Euclidean norm of the corresponding vector. Particularly,

$$\Pi_k(u_1^*, \ldots, u_n^*) = q_k^T \Phi(0)s_k(0) + \int_0^\infty \left[ ||u_k^*||^2 - 2u_k^o u_k^* + q_k^T B_k \Phi(t) Ne^{rt} u_k^* o_k^T \right.$$ 

$$- q_k^T \Phi(t)e^{-rt} \left( \sum_{j=1}^n B_j (u_j^* - u_j^o) + 2 \sum_{j=1}^n (u_j^* - u_j^o)^T B_j B_k^{-1} u_k^* \right)$$

$$\times e^{-rt} dt$$

(B.7)
Considering (B.6) and (B.7), we have

\[ u_{k-1}^*, u_k, u_{k+1}^* \]

\[ = q_k T \Phi(0) s_k(0) + \int_0^{\infty} -\|u_k - u_k^*\|^2 + \|u_k^*\|^2 - 2u_k^T u_k^* + q_k T B_k \Phi(t) N e^{rt} u_k^* \]

\[ = q_k T \Phi(t) e^{tN} \sum_{j=1}^n B_j (u_j^* - u_j^0) + 2 \sum_{j=1}^n (u_j^* - u_j^0)^T B_j B_k^{-1} u_k^* \]

\[ e^{-rt} dt \]

\[ = \Pi_k(u_1^*, \ldots, u_n^*) - \int_0^{\infty} \|u_k - u_k^*\|^2 e^{-rt} dt \leq \Pi_k(u_1^*, \ldots, u_n^*), \quad (B.8) \]

for every \( u_k \). This completes the proof of our theorem.

**Appendix C. Proof of Theorem 3.2**

Considering (10) and (11), the second equation in (8) becomes,

\[ \Phi_{u_i}(t) = \frac{1}{2} \rho_i^2 q_i(n - 1) N_i \Psi_{u_i}(t) e^{rt} - e^{-rt} \Phi_{u_i}(\infty) = 0, \quad i = 1, \ldots, m. \quad (C.1) \]

Let

\[ \Psi_i(t) = \Phi_{u_i}(t) e^{rt} \quad (C.2) \]

Then (C.1) becomes the following Riccati-type equation,

\[ \Psi_i(t) = -1 + r \Psi_i(t) + \frac{1}{2} \rho_i^2 q_i(n - 1) N_i \Psi_i(t), \quad (C.3) \]

\[ \lim_{t \to \infty} \Psi_i(t) e^{-rt} = 0, \quad i = 1, \ldots, m. \]

Equation (C.3) depends only on time and not on initial conditions. As a result, the time variant closed-loop Nash equilibrium strategy in (9) (i.e. an equilibrium that is a function of time, state and initial conditions) becomes a time-variant feedback Nash equilibrium strategy (i.e. an equilibrium that is a function only of time and state). Furthermore, below we show that (C.3) can be solved analytically which completes the proof of our theorem.

Through the substitution

\[ \Psi_i(t) = a_i + V_i(t) \quad (C.4) \]

\[ a_i = \frac{-r - b_i}{\rho_i^2 q_i(n - 1) N_i} \quad \text{and} \quad b_i = \left[ r^2 + 2 \rho_i^2 q_i(n - 1) N_i \right]^{1/2} \]
the Riccati Eq. (C.3) can be transformed into the first order linear differential equation

$$\dot{V}_i(t) - b_i V_i(t) = \frac{-1}{2} \rho_i^2 q_i(n-1) N_i.$$  

Solving (C.5), and considering the boundary condition given in (C.3), we obtain

$$V(t) = e^{h t} + c_i,$$  

where

$$c_i = \frac{\rho_i^2 q_i(n-1) N_i}{2 b_i}.$$

References


