Dynamic Strategic Pricing and Speed of Diffusion

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Abstract. Defining speed of diffusion as the amount of time it takes to get from one penetration level to a higher one, we introduce a dynamic model in which we study the link between pricing policy, speed of diffusion, and number of competitors in the market. Our analysis shows that, in the case of strategic (oligopolistic) competition, the speed of diffusion has an important influence on the optimal pricing policy. In particular, we find that higher speeds of diffusion create an incentive to strategically interacting firms to lower their prices.


1. Introduction

The analysis of the diffusion of new products (innovations) in durable goods markets is a central theme of marketing research. Diffusion research focuses typically on three distinct issues: (i) modeling of the adoption process and the social interactions between adopters and nonadopters of a product; (ii) quantitative and empirical estimation of the adoption rates and hence the speed of diffusion; (iii) analysis of the impact of marketing mix variables and strategies on the diffusion of innovations.

Theoretical modeling of the adoption process has a long tradition in marketing and by now the most prominent model for consumer durables is the Bass model (see Ref. 1). According to this model, adoption

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is the result of two interacting forces, innovation and imitation. Innovation refers to all those phenomena that cause a nonbuyer to adopt a new product without interacting with early adopters. Imitation refers to the effect that causes a nonbuyer to adopt the product as a result of social interactions with early adopters (word of mouth effect). Innovation and imitation provide enough flexibility for the Bass model so that it is capable of explaining many empirical patterns of adoption processes.

While marketing researchers need to understand the diffusion process of a new product, it is also central to them to know how marketing variables such as price, advertising, and product quality affect the adoption process. This has led to a vast literature on normative diffusion models, which predict optimal price and/or advertising strategies over the product life cycle for monopoly or competitive situations. The general theme in these papers is to incorporate marketing mix variables into the diffusion model and derive optimal strategies over the product life cycle. It is an open issue in the marketing literature, however, how the speed of diffusion influences the price and/or advertising strategy chosen by a firm.

The aim of this paper is to present a theory that explores in detail the impact of the speed of diffusion on the pricing strategy of a firm. We consider a consumer durable that is produced and sold in an oligopolistic market. The product is differentiated but characterized by a constant speed of diffusion during the product life cycle. Since firms set prices over a fixed planning horizon, the problem is formulated as a differential game and optimal prices are derived as closed-loop equilibrium strategies. It turns out that the speed of diffusion has an important impact on the equilibrium pricing strategies set by the firms. In particular, we show that, when firms are engaged in strategic competition, a higher speed of diffusion causes the individual firm to decrease the price. This is a surprising result that can be given the following intuitive explanation. A higher speed of diffusion implies that, independently of a firm's strategy, the product life cycle becomes shorter. In order to capture a larger share of the market, in a situation where the life span of the product becomes shorter, an oligopolistic firm needs to set prices aggressively. A lower price is the strategic response to this.

In the limiting case when the number of firms in the market is one (i.e., a monopoly market) any strategic interactions disappear and a higher speed of diffusion does not have any impact on the pricing policy.

Our paper is organized as follows. In Section 2, we define the concept of the speed of diffusion and present an oligopolistic dynamic pricing model for a consumer durable that incorporates this concept. In Sections 3 and 4, we

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4Among the literature on normative diffusion models, we mention Refs. 2–19.
present our results. We start our analysis with the discussion of an oligopolistic market structure in which firms strategically interact and consider next the case of a single producer. For both cases, we state first the results and then give intuitive marketing interpretations. Finally, Section 5 summarizes our findings, discusses possible extensions of our model, and gives directions for future research.

2. Model

Our aim is to develop a model that relates the pricing policy of a firm to the speed of diffusion for a new consumer durable. For that, we define first a measure of the speed of diffusion.

Speed of Diffusion. The literature on diffusion models offers several measures of the speed of diffusion. Generally, speed is measured as the distance traveled per unit of time. Applying this concept to the speed of diffusion for a consumer durable, we can associate (consistently with marketing literature, see for example Ref. 20 or Ref. 21) two different levels of market penetration with distance and the remaining length of the product life cycle with time. Here, the time units are related to the date where the existing product has to be replaced by a new one. $^5$

Using these concepts, we define speed of diffusion as the percentage increase in the number of adopters (or decrease in the number of nonadopters) resulting from a one percent decrease in the remaining length of the product life cycle. If we denote with $M(t)$ the number of nonadopters and with $z(t)$ the remaining length of the product life cycle at time $t$, the speed of diffusion $\alpha$, can be defined as

$$\alpha = \frac{dM(t)/M(t)}{dz(t)/z(t)},$$

where $dM(t)$ measures the change in the number of nonadopters and $dz(t)$ the change in the remaining length of the product life cycle. As an example, $\alpha = 2$ means that, as the product lifetime shortens by one percent, the number of nonadopters decreases by two percent.

The concept of speed of diffusion as introduced in equation (1) is independent of the market structure and a specific diffusion process. Therefore, next we need to incorporate both, diffusion dynamics and the level of competition into this framework.

$^5$In most of the definitions of the speed of diffusion, regular calendar time is used for the time dimension of the speed (see e.g. Refs. 21 and 22). Here, we deviate from this definition to emphasize that, from a firm’s point of view, the important time scale is the date when the old durable has to be replaced by a new one.
Market Structure and Diffusion Process. We consider a market for a heterogeneous consumer durable with no repeat purchases in which \( n \) oligopolistic firms supply products. The total size of the market is constant and given by \( m \). Let \( N_i(t) \) be the number of adopters of the firm \( i \), at time \( t, i = 1, \ldots, n \). Hence, the remaining market potential for every firm \( i \) at time \( t \) is given by \( m - \sum_{i=1}^{n} N_i(t) \). The remaining market potential is equivalent to the number of nonadopters for which all rival firms in the market compete. If we neglect competitive interactions of rival firms for a moment, the adoption process of firm \( i \) can be described best by an exogenous diffusion model. Concentrating on that stage of the product life cycle where the product is already well established, net sales of firm \( i \), \( \dot{N}_i(t) \equiv dN_i/dt \), and hence the diffusion at time \( t \) can be modeled as

\[
\dot{N}_i(t) = f_i \left[ m - \sum_{i=1}^{n} N_i(t) \right],
\]

(2)

where \( f_i \) is the exogenous firm specific rate of adoption. Equation (2) represents the simplest form of a diffusion model in which the dominating forces are the saturation effects. As specified in (2), the diffusion process is exogenous with respect to the firms’ marketing actions. It is well understood in the marketing literature, however, that the rate of adoption depends critically on marketing mix variables. This leads us to introduce a price-dependent adoption rate \( f_i \). Denote with \( p_i(t) \) the price charged by firm \( i \) at time \( t \). Allowing for strategic price interactions among rival firms results in a generalized diffusion model given by

\[
\dot{N}_i(t) = f_i(p_1, \ldots, p_n) \left[ m - \sum_{i=1}^{n} N_i(t) \right],
\]

(3)

where the adoption process is governed by competitive price interactions and where we assume that the initial level of adopters \( N_i(0) = N_{i0} \) is given. From the specification (3), it is clear that the adoption rate \( f_i \) of firm \( i \) depends on the firm’s own price as well as on the prices of the competitors \( p_j, j \neq i \). To keep the analysis as simple as possible, we specify the price driven part of the adoption rate as

\[
f_i = a - bp_i(t) + \gamma \sum_{j=1, j \neq i}^{n} (p_j(t) - p_i(t)), \quad i = 1, \ldots, n,
\]

(4)

where \( a, b, \gamma \) are given constants. The price functions (4) specify that the likelihood of adoption of firm \( i \)’s product is increased when either the firm
lowers its price or charges a price that is below the prices of the competitors.

With given sales and prices, revenues \( R_i(t) \) of firm \( i \) can be calculated as

\[
R_i(t) = \dot{N}_i(t)(p_i(t) - c),
\]

where \( c \) are the unit production costs which are assumed to be constant.

If firms are now engaged in multiperiod optimization, they need to set a pricing strategy over the fixed planning horizon \([0, T]\) so as to maximize the discounted stream of profits subject to demand and strategic constraints. The demand constraint is given by the adoption process and is specified by equation (3). The strategic constraint relates to the strategies chosen by the rival firms in the market. In principle, there are two contrasting concepts for modeling the strategic constraints. Firms can use open-loop or closed-loop strategies. If they choose open-loop strategies, they design their prices as time-profiles at the beginning of the game and commit themselves to stick to these preannounced time functions for the remaining planning horizon. If they choose closed-loop strategies, they design their prices as decision rules that depend on the current level of the state variable of the system, which in our case is the level of adoption.

Formally, the differential game problem of a firm \( i \) can be stated as follows:

\[
\max_{p_i} \Pi_i = \int_0^T \dot{N}_i(t)[p_i(t) - c]e^{-rt} dt,
\]

s.t. \[ \dot{N}_i(t) = \left[ a - b p_i(t) + \gamma \sum_{j=1, j \neq i}^n (p_j(t) - p_i(t)) \right] \left( m - \sum_{i=1}^n N_i \right), \]

\[ N_i(0) = 0. \]

We assume that the initial level of adoption for any firm is zero and denote with \( r \) the nonzero discount rate.

The pricing game (6) is a standard problem within the class of normative diffusion models, that was already discussed in the literature (Refs. 19 and 23–24).

Incorporation of the Speed of Diffusion. Our definition of the speed of diffusion given in equation (1) can now be used to relate the general pricing game specified in (6) to an explicit modeling of the speed of diffusion. According to our discussion above, \( M(t) \) is defined as
\[ M(t) = m - \sum_{i=1}^{n} N_i(t). \]  

(7)

It is easy to show that, for a given constant \( \alpha \), our definition of the speed of diffusion introduces the following relationship between the number of nonadopters and the remaining length of the product life cycle

\[ z(t) = M(t)^{1/\alpha}. \]  

(8)

Equation (8) relates the current market potential \( M(t) \) to the remaining length of the product life cycle through the speed of diffusion. For example, if the diffusion speed is equal to 2, then a saturation level of 25\% is reached exactly in the middle of the product life cycle, \( z(t) = 0.5 \).

If we apply now the transformation (8) to our new product-pricing problem (6), we are in a position to link diffusion speed to optimal pricing strategies of oligopolistic firms and analyze its impact. If we now normalize the total market size \( m \) to one, the application of (8) onto the pricing game (6) results in the following differential game;

\[
\begin{align*}
\max_{p_i} \Pi_i &= \int_0^T (p_i(t) - c_i) \left[ a - bp_i(t) + \gamma \sum_{j \neq i} (p_j(t) - p_i(t)) \right] z(t)^{\alpha} e^{-rt} dt, \\
\text{s.t.} \quad \dot{z}(t) &= -\frac{z(t)}{\alpha} \sum_{i=1}^{n} (a - bp_i), \\
\quad z(0) &= 1.
\end{align*}
\]  

(9a, 9b, 9c)

It is now our aim to solve this game and analyze the relationship between the diffusion speed and the equilibrium pricing strategies set by the firms.

As mentioned above, firms can choose between two alternative strategy concepts when setting their prices. In this paper, we only allow firms to set closed-loop prices. Hence, in equilibrium, each firm sets a price that is calculated on the basis of a decision rule. These decision rules relate the current price of each competitor to the current level of \( z(t) \) and therefore to \( M(t) \) and \( \alpha \). Thus, equilibrium prices do depend on the speed of diffusion and the level of adoption.

\footnote{It should be pointed out that our model has two different forces that influence the speed of diffusion. One is given by the price response function in (4) and is endogenous to the problem; the other one is given by the exogenous speed \( \alpha \). Here, our attention is focused on how this exogenous speed influences the pricing strategies of the rival firms.}
3. Closed-Loop Equilibrium Pricing Strategies

When firms set closed-loop equilibrium prices, they design their strategies as decision rules that relate the current level of the state variable to the current level of the control variable. In our game (9), the remaining length of the product life cycle \( z(t) \) is the state variable; hence, the closed-loop decision rules are functions of \( z(t) \). Since through (8) there is a one-to-one relationship between \( z(t) \) and \( M(t) \), we can argue that the pricing rules are a function of \( M(t) \) as well. Of course, the closed-loop decision rules also depend on the parameters of our model. For the pricing game (9), we have two specific parameters in which we are interested, the level of the speed of diffusion \( \alpha \) and the number of competitors in the industry \( n \). Since the equilibrium prices do depend on those parameters, we can evaluate their impact in terms of a comparative static analysis.

Since closed-loop strategies do depend on \( M(t) \), they really capture strategic interactions among firms. To see this, consider the state dynamics (3). Suppose that one of the firms finds it optimal to decrease the price. Through the diffusion equation (3), this causes additional consumers to adopt the product from this firm. As a consequence, the level of \( M(t) \), the current number of nonadopters decreases. If firms use closed-loop strategies, this change in the level of the remaining market potential causes the rival firms to adjust their prices. Hence, the closed-loop decision rules characterize action and reaction behavior present in the strategic interactions.

From a managerial point of view, closed-loop strategies are therefore more attractive and, because they are decision rules, they supply more useful information to managers. However, closed-loop strategies are mathematically more difficult to obtain. Fortunately, for this problem we are able to find the closed-loop strategies and therefore the equilibrium pricing policies. This allows us to analyze the influence of the speed of diffusion and other parameters on pricing. Not only our model admits an explicit characterization of the closed-loop decision rules, but also a derivation of a differential equation that characterizes the time profile of these pricing rules. Both results are presented in our first two theorems.

**Theorem 3.1.** The dynamic pricing game (9) has a symmetric Nash equilibrium in closed-loop strategies. The time trajectory or control path of this pricing strategy is the solution of the following Riccati differential equation:
\[
\dot{p}(t) = \frac{1}{A} \left\{ \left[ A(p - c) - (a - bc) \right][r - (p - c)\left(1 - \frac{c}{A}\right)(n - 1)b \\
+ [(1 - \frac{c}{A})(a - bc - \gamma(p - c)) + n(a - bp)] \
- b(p - c)(a - bp) \right\}, \quad (10a)
\]
\[
p(T) = \frac{a + c(A - b)}{A}. \quad (10b)
\]

**Proof.** See the Appendix, Section 6.

It is obvious that, for every closed-loop pricing rule, we can calculate a price path that is a function of time. It should be pointed out, however, that this time path is in general not identical to the open-loop pricing strategy. For our model, however, it turns out that, in the case \( \alpha = 1 \), the symmetric closed-loop price is degenerate in the sense that it corresponds with the open-loop equilibrium price. For \( \alpha \neq 1 \), the two are different and therefore we can derive the state-dependent closed-loop decision rules.

**Theorem 3.2.** The dynamic pricing game (9) has a symmetric Nash equilibrium in closed-loop strategies. The decision rule \( p^* = p^*(M, t; \alpha, n) \) is the solution of the following backward differential equation:

\[
p'(M) = \frac{n(a - bp)M}{A} \left\{ \left[ A(p - c) - (a - bc) \right][r - (p - c) \\
\times \left(1 - \frac{c}{A}\right)(a - bc - \gamma(p - c)) + n(a - bp)] \
- b(p - c)(a - bp) \right\}, \quad (11a)
\]
\[
p(1) = p_1, \quad (11b)
\]

where

\[
p_1 = P(t; \alpha, n)|_{t=0} \quad (12)
\]

and \( P(t; \alpha, n)|_{t=0} \) is the solution of equation (10).

**Proof.** See the Appendix, Section 6.

The decision rules (11) specify the equilibrium prices as a function of the level of nonadopters \( M(t) \), the speed of diffusion \( \alpha \), and the number of competitors \( n \) in the market. To illustrate the results in Theorems 3.1 and 3.2 graphically, let us use the following parameter specifications:

\[
a = 1, \quad b = 1, \quad c = 0.5, \quad r = 0.1, \quad \gamma = 1, \quad n = 4. \quad (13)
\]

The results are plotted in Figures 1 and 2. Figure 1 illustrates equilibrium prices as a function of the fraction of adopters \( 1 - M(t) \) for two different levels of the diffusion speed; therefore, it illustrates Theorem 3.2. From
this, we see that, as the fraction increases, the equilibrium prices decrease. Hence, firms are following an intertemporal price discrimination strategy. Early adopters have to pay a higher price than later adopters. This figure illustrates also that, at the same level of adoption, a higher speed of adoption results in a lower price. A detailed discussion of this issue is presented in the next section.

Figure 2 illustrates the time profile of the equilibrium price, again for two different levels of the diffusion speed. It shows that prices decrease over time, starting at a lower level for a higher diffusion speed, but are always above static levels, which are given by the terminal prices. This again is an immediate effect of saturation effects. Additionally, we see that firms in this market keep their price level constant during most of the
planning horizon. As the terminal time approaches, however, prices are decreased to meet the terminal condition.

As we see from this first discussion, the dynamic pricing game delivers expected results. In Section 4, we are going to explore these results in detail and discuss their marketing implications. According to our focus, we are interested primarily in the relationship between the diffusion speed and competitive pricing. To fully characterize our pricing results, we either make use of the closed-loop decision rule (11) or the differential equation (10), which holds for the time trajectory of optimal prices.

4. Influence of the Speed of Diffusion on the Pricing Strategy

The theorems presented in the preceding section demonstrate that the optimal prices do depend on the speed of adoption. While in principle any speed of adoption is possible, we restrict our analysis to the case $\alpha \geq 1$. This is motivated by the sufficient optimality conditions. Only, if $\alpha \geq 1$, we can show that the maximized Hamiltonian satisfies the necessary concavity conditions.

When the diffusion process is characterized by $\alpha = 1$, we are in a situation where the speed of adoption is in a one-to-one relationship with the remaining time of the product life cycle. Hence, the two variables $M(t)$ and $z(t)$ are not distinguishable. This case corresponds also to traditional analyses carried out in dynamic marketing mix diffusion models. When $\alpha > 1$, we can analyze the impact of the diffusion speed on pricing.

For the moment, however, we are not interested in the impact of $\alpha$ on the price in a strategic environment, but will look at the limiting case where $n = 1$. In such an environment, it is already known for a long time that, with dominating saturation effects, the optimal monopoly pricing strategy consists of a decreasing price over time, i.e., a skimming or inter-temporal price discrimination strategy. With $n = 1$, equation (10) reproduces this result. Secondly, we get that the dynamic price is above the static one. This is known also from existing papers in the literature. What is not known, however, is the impact of the diffusion speed on the dynamic price. Based on the results presented in Section 3, we arrive at the following proposition.

**Proposition 4.1.** In the limiting case when the consumer durable market is supplied by a monopoly producer, the optimal dynamic pricing strategy is independent of the speed of diffusion.

**Proof.** See the Appendix, Section 6.
The result of Proposition 4.1 is interesting. Only if a single producer controls the market, the speed of diffusion, as defined in Section 2, has no impact on the level as well as on the time profile of the profit-maximizing price. Although surprising, this result has a very intuitive interpretation. A monopoly producer is aware of the fact that the entire market potential can only be exhausted by itself. Therefore, no matter what the speed of diffusion is, it need not worry that potential rivals break in and steal away some customers. This awareness causes it to set a single price even when different exogenous speeds of diffusion prevail.

Next, we are interested in the case with oligopolistic competition. Here, we want to shed light on two independent questions. What does the optimal pricing policy over time look like? And how does the diffusion speed affect the equilibrium price, when firms are engaged in strategic competition? We will address these questions now.

**Proposition 4.2.** In the case of strategic competition, the time trajectory of the symmetric closed-loop Nash equilibrium pricing strategy decreases over time. Moreover, symmetric equilibrium prices are lower when the speed of diffusion is higher.

**Proof.** See the Appendix, Section 6.

A decreasing time profile of equilibrium prices is the expected result and is illustrated in Figure 2. Many pricing diffusion models arrive at the same conclusion for open-loop strategies. It must be emphasized, however, that our result holds for closed-loop strategies. In that respect, it generalizes many existing papers. What is surprising is the effect of the diffusion speed on the level of prices. In the case with oligopolistic competition, prices decrease as the speed of diffusion increases. There is an intuitive explanation for this result. A higher exogenous speed of diffusion implies that, independently of a firm’s strategy, the product life cycle becomes shorter. In order to capture a larger share of the market, in a situation where the life span of the product becomes shorter, an oligopolistic firm needs to set prices aggressively. A lower price is the outcome of this behavior. Moreover, it should be remembered that, with \( \alpha \neq 1 \), the closed-loop prices do depend on the level of adopters (they are not degenerate as in the case of \( \alpha = 1 \)), so that the speed of diffusion must have an impact. This is stated in our next proposition.

**Proposition 4.3.** The symmetric closed-loop Nash equilibrium pricing strategy decreases with the number of adopters.
Proof. See the Appendix, Section 6.

The last results gives advice to marketing managers who implement the equilibrium decision rule when setting an optimal price. All they need to know is the level of adopters to calculate the optimal price. As this level increases, the firms should lower their prices in order to be attractive for nonadopters to buy the product in the next period. This holds true for the monopoly as well as the oligopoly market. In both market structures, the decision rules are driven by the existence of saturation effects as the dominating force on the demand side.

So far, we have looked only at markets with a fixed number of firms. One of the characteristics of durable goods markets is the observation that, as the product becomes mature, new firms enter the market. Therefore, we will explore the consequences of entry on prices next. It should be pointed out, however, that entry in our model is treated like an exogenous change in the number of firms who compete in the market. Alternatively, one could model entry as a strategic game between incumbent firms and potential entrants. This would require having two types of games: a pricing game for the existing firms in the market and an entry game for the potential entrants. However, it is outside the scope of this paper to explore this scenario any further.

Proposition 4.4. As the number of rivals in the oligopolistic durable goods market increases, the prices decrease.

Proof. See the Appendix, Section 6.

The result in Proposition 4.4 is referred to frequently in the economics literature as quasicompetitive behaviour; i.e., as the number of competitors in the market increases, the prices decrease and approach the level of costs. From a marketing point of view, it implies that the power to set prices shrinks with the number of rival firms. This is an expected result that can be found in static as well as dynamic models.

5. Conclusions

In this paper, we present a theory that defines the concept of the speed of diffusion as the percentage increase in the number of adopters (or decrease in the number of nonadopters) resulting from a one percent decrease in the remaining length of the product life.
Using this concept, we explore the impact of the speed of diffusion on the pricing policy of a firm that sells a consumer durable either in a monopoly or in an oligopoly market. For the limiting case when the durable is sold by a monopolist, we find that the speed of diffusion has neither an influence on the level of the price nor on its rate of change. In a monopoly market, the single seller can be sure that the entire market demand is its own so that, with a skimming strategy, it eventually will exhaust the market. If the durable is sold in an oligopoly market, the speed of diffusion has a significant impact on the pricing policy adopted by the rival firms. In particular, as the speed of diffusion increases, it is optimal to set lower prices. These results have strong managerial implications. Our theory predicts that monopolistic sellers need not worry about all aspects of the product life cycle when designing a pricing policy. The dynamic price should account for the decrease in the remaining market potential, but not for the speed of diffusion. The situation is quite different when firms are engaged in strategic competition. In this setting, a marketing manager not only needs to observe the actions of the rival firms but also the external forces that cause the speed of diffusion to change. Hence, demographic and other data are important to design an optimal marketing strategy.

It should be emphasized that our model is very simple and ignores a lot of important characteristics of real-world durable goods markets. One such feature is a more general product life cycle. In this model, we concentrate on that phase of the product life cycle in which there are only saturation effects. It would be an interesting avenue for future research to analyze if our findings carry over to more dynamic diffusion effects such as innovation and imitation as well. A second drawback of the present model is its focus on pricing policies. However, any sensible marketing strategy needs to design an optimal marketing mix strategy for a firm. Hence, it would be an interesting challenge for future research to study the impact of the speed of diffusion on other marketing variables such as advertising, product quality, or product design. Finally, we want to point out that our theory can be used to derive testable hypotheses. It would be interesting to find out what the data say about the impact of the speed of diffusion on the pricing strategy of a firm.

6. Appendix: Proofs

Proof of Theorem 3.1. We will solve problem (9) by using dynamic optimization techniques; see Ref. 25 or 26, for example. Define the
current-value Hamiltonian of firm \( i \) as follows:

\[
H_i = (p_i - c) \left[ a - b p_i + \gamma_i \sum_{j=1, j \neq i}^{n} (p_j - p_i) \right] z^\alpha - \lambda_i z / \alpha \sum_{i=1}^{n} (a - b p_i) \tag{14}
\]

where \( \lambda_i \) is the current-value adjoint variable.

Considering (14) and assuming that the firms use closed-loop strategies, the necessary conditions become

\[
\frac{\partial H_i}{\partial p_i} = \begin{cases} 
  a - b p_i + \gamma_i \sum_{j=1, j \neq i}^{n} (p_j - p_i) - [b + \gamma (n - 1)] (p_i - c) \\
  z^\alpha + \left( \frac{b}{\alpha} \right) \lambda_i z = 0; 
\end{cases} \tag{15}
\]

the adjoint equations are

\[
\dot{\lambda}_i = r \lambda_i - \frac{\partial H_i}{\partial z} - \sum_{j=1, j \neq i}^{n} (\partial H_i / \partial p_j) (\partial p_j / \partial z); 
\]

therefore,

\[
\dot{\lambda}_i = \left[ r + \frac{1}{\alpha} \sum_{i=1}^{n} (a - b p_i) \right] \lambda_i - \alpha (p_i(t) - c) \\
	imes \left[ a - b p_i(t) + \gamma_i \sum_{j=1, j \neq i}^{n} (p_j(t) - p_i(t)) \right] z^{\alpha - 1} \\
- \sum_{j=1, j \neq i}^{n} [(p_i - c) z^\alpha \gamma_j + (z \lambda_i / \alpha) b] \left[ (1 - \alpha) b \lambda_j / \alpha (2b + N \gamma_j) \right] z^{\alpha - 1}, \tag{16}
\]

with the boundary condition

\[
\lambda_i(T) = 0. \tag{17}
\]

In (16), we made use of

\[
\partial p_j / \partial z = \left[ (1 - \alpha) b_j / \alpha (2b + N \gamma_j) \right] \lambda_j z^{-\alpha}. 
\]

Since all the firms face symmetric conditions, a symmetric equilibrium, i.e.,

\[
p_i = p, \ \forall i = 1, \ldots, n, \tag{18}
\]
satisfies the necessary conditions. Then, obviously, 
\[ \lambda_i = \lambda, \quad \forall i = 1, \ldots, n. \] (19)
Considering (18) and (19), we obtain from (15) that the closed-loop Nash equilibrium strategy of the differential game (9) is 
\[ p = \frac{1}{A}[(b/\alpha)\lambda z^{1-\alpha} + a + (A - b)c], \] (20)
where 
\[ A = 2b + \gamma (n - 1) \] (21)
and where \( \lambda \) satisfies
\[ \dot{\lambda} = [r + (n/\alpha)(a - bp)]\lambda - \alpha(p - c)(a - bp)z^{\alpha - 1} - [(p - c)\gamma + (b/\alpha)\lambda z][(a - 1)(1 - a)b/A\alpha] \lambda z^{-\alpha}. \] (22a)
\[ \lambda(T) = 0. \] (22b)
If we differentiate (20) with respect to \( t \), we obtain
\[ \dot{p} = \frac{1}{A}(b/\alpha)\dot{\lambda} z^{1-\alpha} + [(1 - a)b/a]\dot{\lambda} z^{-\alpha}. \] (23)
Substitution of (9a), the necessary condition (20), and the adjoint equation (22) into equation (23) results in a Riccati differential equation in the price \( p \). The boundary condition is a result of substituting the boundary condition (22) into (20).

**Proof of Theorem 3.2.** Equation (11) is an autonomous first-order differential equation in the price \( p \) that, together with the terminal condition, determines the equilibrium price for any \( \alpha \). Furthermore from the smoothness of the right-hand side of equation (11), we conclude that, for every \( \alpha \geq 1 \), there exists a unique solution of the differential equation (11). Let the unique solution, for every \( \alpha \geq 1 \), for the symmetric equilibrium be \( P(t; \alpha, n) \), where \( 0 \leq t \leq T \) and let 
\[ p_1 = P(t; \alpha, n)|_{t=0}. \] (24)
Considering (20), we can assume that, for \( \alpha \neq 1 \), we have a pricing decision rule as a function of \( M \). In this case, we can substitute the relationship 
\[ \dot{p}(t) = p'(M)\dot{M}(t), \]
where 
\[ \dot{M} = -n(a - bp)M, \]
in the Riccati equation, and taking into account the initial condition (24), we arrive at equation (11) and, as a consequence, with a decision rule of the form \( p^*(M, t; a, n) \).

**Proof of Proposition 4.1.** Consider the limiting case \( n = 1 \). Then, equation (10) becomes

\[
\begin{align*}
\dot{p} &= (1/2b)[2b(p - c) - (a - bc)]r - b(p - c)(a - bp), \\
p(T) &= (a + cb)/2b.
\end{align*}
\tag{25a, b}
\]

Hence, the proposition follows by considering (25). □

**Proof of Proposition 4.2.** Since the differential equation (10) is autonomous and the right hand side is quadratic in \( p \), it follows that any solution that satisfies the terminal condition must be monotonic in \( t \).

Moreover, we have

\[
\dot{p}(T) = -(1/A^2)b(a - c)\gamma\left(1 - \frac{b}{A}\right) < 0.
\tag{26}
\]

Since any solution to (10) is monotonic, we can conclude that the prices are decreasing over the entire planning period \([0, T]\) or

\[
\dot{p}(t) < 0, \quad \forall t \in [0, T],
\tag{27}
\]

and this proves the first part of the proposition. To prove the second part, we note first that, because of

\[
\ddot{p}(T) = \left[\dot{p}(T)/A\right]\left\{Ar + (b^2(a - bc))/A + (a - bp(T))(An - b)
\right.
\]

\[
\left.-[(1 - \alpha)\gamma/\alpha A](n - 1)b(a - bc)\right\} < 0,
\tag{28}
\]

prices are concave functions in a neighborhood of the terminal time \( T \). Next, observe that

\[
\ddot{p}(T)|_{\alpha=1} - \ddot{p}(T)|_{\alpha>1} = \left[\dot{p}(T)/A\right][(1 - \alpha)\gamma/\alpha A](n - 1)b(a - bc) > 0.
\tag{29}
\]

Hence,

\[
\ddot{p}(T)|_{\alpha=1} > \ddot{p}(T)|_{\alpha>1}
\tag{30}
\]

and pricing strategies with a higher diffusion speed have a higher curvature in a neighborhood of \( T \) and hence lie below them. However, this together with the property that \( \partial \dot{p}/\partial \alpha > 0 \) implies that our equilibrium
prices are lower the higher the speed of diffusion is and this completes the proof of our proposition.

Proof of Proposition 4.3. Considering (27) and the relation \( p'(M) = \frac{\dot{p}(t)}{\dot{M}(t)} \) and since \( \dot{M}(t) < 0 \) [see (7)], we conclude with

\[
p'(M) > 0; \tag{31}
\]
therefore, the price increases with the number of nonadopters or decreases with the number of adopters for any \( n \geq 1 \).

Proof of Proposition 4.4. The time profiles of the equilibrium prices are governed by a Riccati differential equation (equation (10)). Given the terminal condition, it is possible to solve for the optimal prices explicitly. We get \( p(t; n, \alpha) \). It can be shown by differentiation that, as \( n \) increases, \( p \) decreases to the competitive level.

References


