Generalized Zero Sets Location and Absolute Robust Stabilization of Continuous Nonlinear Control Systems*

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An analytic method for finding the location of the generalized zero set of a vector-valued function is applied here to find the complete feasible set of sectors of nonlinearities which allows absolute robust stability of a Lurie type system with linear part under uncertainty conditions, according to the Popov criterion.

Key Words—Control system design; nonlinear control systems; robust control; stability; zeros.

Abstract—We describe an analytic method for finding the location of the generalized zero set of a vector-valued function which depends on m real variables and (n + k) complex parameters. The method is applied to a robust design problem of a nonlinear Lurie type continuous-time system, with the linear part under uncertainty conditions. We find the complete feasible set of sectors of the nonlinearities which allows robust absolute stability of the system, according to the Popov criterion. Illustrative numerical examples are provided.

1. INTRODUCTION

Various problems which arise in system theory can be reduced to the problem of locating the generalized zero set, defined as follows.

A generalized zero set. Given an open set G in $\mathbb{R}^m$, $m \geq 1$, closed sets $Q_1, \ldots, Q_n$, $P_1, \ldots, P_k$ in $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ with $Q = Q_1 \times \cdots \times Q_n$, $P = P_1 \times \cdots \times P_k$ and a continuous function $f : Q \times P \times G \rightarrow \mathbb{R}^d$, the generalized zero set of $f$ relative to $Q$, $P$ and $G$ is defined by

$$M = \{s \in G : f(q, p, s) = 0 \text{ for some } q \in Q \text{ and all } p \in P\}.$$

If we let $f_{q,p}(s) = f(q, p, s)$, $q \in Q$, $p \in P$, $s \in G$, then

$$M = \bigcap_{p \in P} \bigcup_{q \in Q} f_{q,p}^{-1}(0).$$

In other words the set $M$ can be written as

$$V_p = \bigcup_{q \in Q} f_{q,p}^{-1}(0), \quad p \in P,$$

where $V_p$ are zero sets (Fruchter, 1988; Fruchter et al., 1987a, 1991a, b) recently developed into a tool and used in solving problems in control theory. Establishing an algorithm to finding $M$ makes more applications possible in a wide variety of areas.

Walach and Zeheb (1982) consider this problem for $m = 2$ and $G = \mathbb{R}^2$, and when $f(q, p, s)$ is a complex-valued function, holomorphic in $q$ and $p$ and a polynomial in the complex variable $s = s_1 + js_2$, and when each $Q_i$ and $P_j$ is either a piecewise-smooth simple arc or a Jordan domain bounded by a piecewisely-smooth curve. They were able to find necessary conditions for the boundary of $M$ and thereby to develop an algorithm for determining $M$ in some cases. Their results have already been instrumental in control theory.

Here, we derive a new algorithm for locating $M$ for continuously differentiable vector-valued functions $f$ and quite general parameter spaces $Q$ and $P$.

The method is applied to a problem of robustness of absolute stability of Lurie type (Lurie, 1954) continuous nonlinear systems, with linear parts under uncertainty conditions. By uncertainty we mean that the transfer function of the linear part may have as coefficients parameters which may take values in a given set. We find all the feasible set of sectors of nonlinearities, for which absolute robust stability of the system under uncertainty conditions in the
linear part is ensured, according to the Popov criterion (Popov, 1961). From the engineering standpoint, such a result will provide the designer of such a system with an excellent design flexibility, in choosing the optimal robust sector considering design constraints.

A partial solution to this problem was given in Siljak’s work (Siljak, 1969a, b, 1989). Siljak’s method gives sufficient conditions for absolute robust stability, according to Popov criterion, for a restricted class of transfer functions, with respect to the form of the appearance of the uncertainty parameters. This is the price of using Kharitonov type solution by imbedding polytopes of dimensions equal to the number of the uncertainty parameters plus two, in the solution set.

The method proposed in this paper gives sufficient and necessary conditions for absolute robust stability with respect to the uncertainties in the linear part, according to Popov criterion. The method is unrestricted to the form in which the uncertainty parameters may appear in the transfer function. The complete set of sectors of the nonlinearities is selected from the two-dimensional set:

\[ \mathcal{P} = \mathbb{R}^+ \times \mathbb{R} : V = \{(k, q) \in \mathbb{R}^+ \times \mathbb{R} : k^{-1} + \text{Re} \left[ (1 + jwq)G(jw, r) \right] > 0, \]  
\[ \forall \omega \geq 0, \forall r \in \mathcal{P} \mathcal{F}, \]

where \( r \) is the vector of all uncertainty parameters and \( \mathcal{P} \mathcal{F} \) denotes the parameter space in which \( r \) may take its values. The set \( V \) is a corresponding two-dimensional zero set of points \((k, q)\) in \( \mathbb{R}^+ \times \mathbb{R} \). It is determined by the analytic methods described in (Fruchter et al., 1987a, 1988, 1991a, b) and illustrated in this paper. The two-dimensional geometric interpretation can be done at any number of uncertainty parameters.

The discrete case of this problem was treated in the previous works (Fruchter et al., 1987b, Fruchter, 1988).

This paper consists of five sections. In Section 2 we present a brief summary on the method of “zero sets location” (Fruchter, 1988; Fruchter et al., 1987a, 1991a, b). In Section 3 we describe a method of finding the generalized zero set \( M \). The method is illustrated by two numerical examples with engineering meaning. In Section 4 we utilize the algorithm in absolute robust stabilization of continuous-time Lurie type nonlinear systems. A numerical example is provided. The paper closes in Section 5 with appropriate concluding remarks.

2. PRELIMINARY RESULTS

We present a brief summary of the method “zero sets location” (Fruchter, 1988; Fruchter et al., 1987a, 1991a, b), including notations and results needed in this paper.

A. The zero set

Definition 1 (Fruchter et al., 1991a). Let \( K = K_1 \times \cdots \times K_n \) be a set in \( \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C} \) \((n \text{ times})\), where each \( K_i \) is a closed set in \( \mathbb{C} = \mathbb{C} \cup \{\infty\} \) whose boundary \( \partial K_i \) is a finite union of piecewise-smooth simple curves and piecewise-smooth closed simple curves. Let \( G \) be an open set in \( \mathbb{R}^m \) and let \( f: K \times G \rightarrow \mathbb{R}^d \cup \{\infty\} \) be a continuously differentiable function. The zero set \( V \) of \( f \) relative to \( K \) and \( G \) is then defined by

\[ V = \{s \in G : \exists A \in K \text{ such that } f(A, s) = 0\}. \]

Lemma (Fruchter et al., 1991a). \( V \) is a closed set relative to \( G \).

B. Zero sets location—Outline of the procedure

There are two principal stages in locating \( V \). In the first stage one finds an \((m - 1)\)-dimensional set \( L, L \subset V \), which contains the boundary of \( V \) relative to \( G \). Next, by picking a point in each of the connected components of \( G \setminus L \) one checks which of these connected components are included in \( V \) and which are outside \( V \). This stage leads to the original problem in a lower dimension, which can be carried out reductively until the final solution is reached. Let \( D_1, \ldots, D_k \) be the connected components of \( G \setminus L \) which are included in \( V \); then

\[ V = \bigcup_{i=1}^{k} D_i \cup L. \]

In order to find the set \( L \) one needs necessary conditions on the boundary of \( V \). In the following we present two theorems which provide such necessary conditions.

C. Necessary conditions on the relative boundary of the zero set

Theorem 1 (Fruchter et al., 1991a). Let \( K, G, f \) and \( V \) be as in Definition 1. Suppose that \( s^0 \) is a point in the boundary of \( V \) relative to \( G \) and \( A^0 = (A^0_1, \ldots, A^0_n) \) is a point in \( K \) such that \( f(A^0, s^0) = 0 \). Suppose that for \( i = 1, \ldots, \ell \), \( 1 \leq \ell \leq n \), \( A^0_i \) belongs to the boundary of a connected component of \( K_i \), having a parametric representation

\[ A_i = A_i(\theta_i), \quad 0 \leq \theta_i \leq 1 \]

and that for \( i = \ell + 1, \ldots, n \), \( 0 \leq \ell \leq n \), \( A^0_i \in \text{int } K_i \).

Suppose that \( \theta_1^0, \ldots, \theta_2^0 \) are numbers in the open intervals \((0, 1)\) such that

\[ A_i(\theta_i^0) = A_i^0. \]
and $x_0^\ell$, $y_0^\ell$ are real numbers such that for
$i = \ell + 1, \ldots, n$, $A_0^\ell = x_0^\ell + jy_0^\ell$.

Let $\delta \in (0, 1)$ be such that $(\theta_0^\ell - \delta, \theta_0^\ell + \delta) \subset
(0, 1)$ for $i = 1, \ldots, \ell$ and
$(x_0^\ell - \delta, x_0^\ell + \delta) \times
(y_0^\ell - \delta, y_0^\ell + \delta) \subset K_i$ for $i = \ell + 1, \ldots, n$.

Denote
$\xi_0^\ell = (\xi_0^0, \ldots, \xi_0^{\ell})$
$= (\theta_0^0, \theta_0^\ell, x_0^0, y_0^0, \ldots, x_0^\ell, y_0^\ell)$
and
$\xi = (\xi_1, \ldots, \xi_r)$
$= (\theta_1, \ldots, \theta_r, x_{e+1}, y_{e+1}, \ldots, x_n, y_n)$
for each point $\lambda \in \mathbb{R}^r$ such that
$|\xi_i - \xi_j| < \delta, \quad i = 1, \ldots, r$.

Then, at points $(A_0^0, s_0)$, where the derivatives of $A_0(\theta_i)$ exist, we have the relations
(i) $f(A_0^\ell, s_0) = 0, \quad i = 1, \ldots, d$
(ii) $\frac{\partial f_i}{\partial \xi_j}(A_0^\ell, s_0) = 0, \quad 1 \leq i_1 < \ldots < i_d \leq r$.

At points where any of the coordinates is $\infty$, the differentiability of $f$ and the conditions (i) and (ii) should be evaluated after suitable changes of variables of the form $z \rightarrow 1/z$ are performed.

Theorem 2 (Fruchter et al., 1991a). Let $K, G$, $f = (f_1, \ldots, f_d)$, $V$, $(A_0^0, s^0)$, $\xi_0 = (\xi_0^0, \ldots, \xi_0^e)$
and $\xi = (\xi_1, \ldots, \xi_r)$, $r \geq 2$, with $A = A(\xi)$ and $A_0 = A(\xi_0)$ be as in Theorem 1, when $d \geq 2$.

Let $k$ and $q$ be integers such that $1 \leq k \leq d - 1$ and $d - k \leq q \leq r - 1$. Suppose that there is a $(m + r - q)$-dimensional neighborhood $N = N_1 \times N_2$ of the point $(\xi_0^1, s^0) = (\xi_0^1, \ldots, \xi_0^e, s^0)$ and a vector-valued function $u: N \rightarrow \mathbb{R}^q$, such that
(i) $u$ is continuously differentiable on $N$
(ii) $u(\xi_{i+1}^1, \ldots, \xi_r^1) = (\xi_{i+1}^0, \ldots, \xi_r^0)$
(iii) $u(A(\xi_{i+1}^1, \xi_r^1), s) = 0$ for every $(\xi_1, s)$
in $N$, where $\xi_1^1 = (\xi_1^0, \ldots, \xi_r^0)$ and $v = (f_{i+1}, \ldots, f_d)$.

Let $g: N \rightarrow \mathbb{R}^k$ be defined by
$g(\xi_1^1, s) = u(A(\xi_1^1, s))$, $s$
where $u = (f_{i+1}, \ldots, f_d)$ and $g = (g_1, \ldots, g_k)$.
Then at points $(\xi_{i+1}^1, s^0)$ we have
$g_j(\xi_{i+1}^1, s) = 0, \quad j = 1, \ldots, k$
$\frac{\partial g_i}{\partial \xi_j}(\xi_{i+1}^1, \ldots, \xi_r^1, s) = 0, \quad q + 1 \leq i_1 < \ldots < i_k \leq r$.

D. An outline of the procedure of determining $L$
Let $f$, $K$ and $G$ be as in Theorem 1 and Theorem 2. For each $\ell$ in $(1, \ldots, n)$, choose a subset of $\ell$ different indices from the set $(1, 2, \ldots, n)$. For each index $i$ in this subset, pick a connected component of $\partial K_i$. Finally, applying Theorem 1 and/or Theorem 2 we can find a set of points $(s_1, \ldots, s_m)$ which is $(m - 1)$-dimensional in $\mathbb{R}^m$. We do the same for all possible choices of $\ell$, $1 \leq \ell \leq n$, of the indices $i_1, \ldots, i_\ell$ from the set $(1, \ldots, n)$ and of connected components of $\partial K_{i_1}, \ldots, \partial K_{i_\ell}$, respectively, and for $\ell = 0$. We denote the union of all the $(m - 1)$-dimensional sets, which are included in $V$, by $L_0$.

Next, consider a finite set of points, say $b_1, \ldots, b_p$, in $\bigcup_{i=1}^n \partial K_i$ which correspond to the points where the derivatives of $A_i(\theta_i)$ do not exist. In the sequel, we will label such points "bad" points. Suppose that $b_1 \in \partial K_1$. By substituting $A_1 = b_1$, $f(A, s) = 0$ reduces to a (vector) equation in the (complex) unknowns $A_2, \ldots, A_n$ and $s \in \mathbb{R}^m$. We apply Theorem 1 and/or Theorem 2 to the new equation at the point $(b_1, A_2, \ldots, A_n)$ and obtain $(m - 1)$-dimensional sets in $\mathbb{R}^m$, included in $V$, in a way similar to the previous procedure for $L_0$. Next we repeat the procedure for $b_2, b_3, \ldots$ up to $b_p$. Denote by $L_1$ the union of all these $(m - 1)$-dimensional sets for $b_1, \ldots, b_p$.

Next, we substitute in $f(A, s) = 0$ two bad points, say $b_n$ for $A_i$ and $b_{nk}$ for $A_{ik}$ such that $i \neq k$, and apply Theorem 1 and/or Theorem 2. We denote by $L_2$ the union of all the sets, which are included in $V$. Next, we choose three bad points, then four bad points, etc., and obtain $L_3, \ldots, L_0$.

Let
$L = \bigcup_{i=0}^p L_i$.
Then, since each $L_i$ is $(m - 1)$-dimensional and included in $V$, so is $L$ and
$\partial \partial V = L \subset V$.

3. GENERALIZED ZERO SETS LOCATION
A. The generalized zero set
Let $Q = Q_1 \times \cdots \times Q_n$ and $P = P_1 \times \cdots \times P_k$
be sets in $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ (n times) and
$\tilde{\mathbb{C}}^k = \mathbb{C} \times \cdots \times \mathbb{C}$ (k times), respectively, where each $Q_i$ and $P_i$ is a closed set in $\mathbb{C} = \mathbb{C} \cup \{\infty\}$. For each $i$ the boundary $\partial Q_i$ and $\partial P_i$ is a finite union of piecewise-smooth simple curves and piecewise-smooth closed simple curves. Let $G$ be an open set in $\mathbb{R}^m$ and let $f: Q \times P \times G \rightarrow \mathbb{R}^d \cup \{\infty\}$ be a continuously differentiable function.
The last assumption means that \( Q \times P \times G \) has an open neighborhood, where \( f(q, p, s) = (f_1(q, p, s), \ldots, f_d(q, p, s)) \) has continuous partial derivatives with respect to all real coordinates \( q_1, q_1', p_1, p_1', s_1, \ldots \), where \( q_i = q_i' + j p_i', \ p_i = p_i' + j p_i'^2, \ q = (q_1, \ldots, q_m) \in Q, \ p = (p_1, \ldots, p_k) \in P \) and \( s = (s_1, \ldots, s_m) \in G \). The generalized zero set of \( f = (f_1, \ldots, f_d) \) relative to \( Q, P \) and \( G \) is then defined by

\[
M = \{ s \in G : \forall p \in P, \exists q \in Q \text{ such that } f(q, p, s) = 0 \}. \tag{1}
\]

In other words, \( s \in M \) if and only if \( f(p, q, s) = 0 \) for some point \( q \) in \( Q \) and every point \( p \) in \( P \).

Our main purpose in this section is to derive an algorithm which will enable us to determine and describe the generalized zero set \( M \) of the vector-valued function \( f \), for certain parameter spaces \( Q \) and \( P \), and open sets \( G \). This objective will be carried out by redefining \( M \) in terms of certain zero sets.

### B. The procedure of determining the generalized zero set

Let \( Q, P, G, f \) and \( M \) be as in Section 3A. In particular it is assumed that \( Q \times P \times G \) has a neighborhood where \( f \) is continuously differentiable. Suppose that \( \tilde{P} \supset P \) is an open set in \( \mathbb{C}^k \) for which \( \tilde{f} \) is continuously differentiable in \( Q \times \tilde{P} \times G \). Let \( \tilde{V} \) be the zero set of \( \tilde{f} \) relative to \( Q \) and \( G \times \tilde{P} \), i.e.

\[
\tilde{V} = \{ (s, p) \in G \times \tilde{P} \subset \mathbb{R}^m \times \mathbb{C}^k : \exists q \in Q \text{ such that } \tilde{f}(q, p, s) = 0 \}. \tag{2}
\]

Excluding points in which one of the coordinates of \( p = (p_1, \ldots, p_k) \in \tilde{P} \) is \( \infty \), we denote points \( (s, p) \) in \( G \times \tilde{P} \) by \( (s, p) = (s_1, \ldots, s_m, p_1, \ldots, p_k, \ldots) \), where \( (p_1 + j p_2) = p_1 \in P \setminus \{ \infty \} \).

Note that the differentiability of \( \tilde{f} \) at points \( (q, p, s) \) where at least one of the coordinates of \( (q, p) \) is \( \infty \) or at points \( (q, p, s) \) for which \( f(q, p, s) = \infty \), is checked by means of a change of variables of the form \( z \to 1/z, \ z \in \mathbb{C} \).

**Lemma 1.** \( \tilde{V} \) is a closed set relative to \( G \times \tilde{P} \). The proof is as in Fruchter et al. (1991a, Lemma II,4), where the idea is to show that \( \tilde{V} \) contains each of its limit points which are in \( G \times \tilde{P} \). This follows from the compactness of \( Q \) (being a closed subset of \( \mathbb{C}^m \)) and the continuity of \( f \).

The set \( \tilde{V} \) has all the properties that the zero set \( V \) considered in Fruchter (1988) and Fruchter et al. (1987, 1991a, b), and presented in Section 2, has, including the fact that \( \tilde{V} \) is closed relative to \( G \times \tilde{P} \). On the boundary of \( \tilde{V} \) relative to \( G \times \tilde{P} \) we have similar conditions to those stated in Theorems 1 and 2. Therefore, in order to find the location of \( \tilde{V} \) in \( G \times \tilde{P} \) we may apply the method briefed in Section 2.

The following proposition is an immediate consequence of (1) and (2).

**Proposition 1.** Let \( Q, P, G, f, M \) and \( \tilde{V} \) be as in 3A. Then

\[
M = \{ s \in G : \exists p \in P, (s, p) \in \tilde{V} \}. \tag{3}
\]

Now, let \( L \) be an \((m-1)\)-dimensional set, as described in Section 2, i.e. \( L \) is included in \( \tilde{V} \) and includes the boundary of the zero set \( \tilde{V} \). Then \( L \) is a finite union of solutions of equations obtained from Theorem 1 and/or Theorem 2. In other words \( L \), and consequently the relative boundary of \( \tilde{V} \), is a finite union of \((m-1)\)-dimensional sets of the form

\[
\{ (s, p) \in G \times \tilde{P} : \varphi_i(p, s) = 0 \},
\]

where the equations \( \varphi_i = 0 \) are derived from the conditions on the relative boundary of \( \tilde{V} \) which Theorems 1 and 2 provide. From the implicit function theorem it follows that \( \varphi_i \) are continuously differentiable real valued functions. Therefore, in many cases, the zero set \( \tilde{V} \) can be expressed as a finite union and intersection of the sets

\[
\tilde{V}_i = \{ (s, p) \in G \times \tilde{P} : \varphi_i(p, s) < 0 \}. \tag{4}
\]

Using Proposition 1 we obtain, in this case, that \( M \) can be expressed by a corresponding finite union and intersection of the sets

\[
M_i = \{ s \in G : \exists p \in P, (s, p) \in \tilde{V}_i \}. \tag{5}
\]

The complement of \( M_i \) in \( G \), denoted by \( S_i \), becomes

\[
S_i = G \setminus M_i = \{ s \in G : \exists p \in P, \varphi_i(p, s) > 0 \}.
\]

Note that

\[
\varphi_i(p, s) > 0
\]

is equivalent to

\[
\varphi_i(p, s) + n > 0 \quad \text{for some } n \in [-\infty, 0).
\]

Denote

\[
h_i(n, p, s) = \varphi_i(p, s) + n \quad \text{for some } n \in [-\infty, 0).
\]

Then \( S_i \) becomes

\[
S_i = \{ s \in G : \exists p \in P, \exists n \in [-\infty, 0) \text{ such that } h_i(n, p, s) = 0 \}.
\]

It is easy to see that

\[
S_i = \bigcup_{\varepsilon < 0} S_i^\varepsilon. \tag{7}
\]
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where

\[ S_f = \{ s \in G : \exists p \in P, \exists n \in [-\infty, \epsilon] \text{ such that } h_1(n, p, s) = 0 \}. \tag{8} \]

The set \( S_f \) is recognized as a zero set, presented in Section 2 of the real-valued function \( h_1 \). In conclusion the sets \( M_i = G \setminus S_i \) and therefore \( M \), can be found by the method presented in Section 2.

In conclusion, \( M \) can be found by the following steps.

Step 1: Choose \( \tilde{P} \) and determine the corresponding set \( \tilde{V} \) [see (2)].

Step 2: Write \( \tilde{V} \) as a finite union and intersection of sets \( \tilde{V}_i \) [see (4)].

Step 3: Write \( M \) as a finite union and intersection of sets \( M_i \) [see (5)].

Step 4: Determine the functions \( h_i \) from \( \varphi_i \) [see (6)] and write the complements \( S_i \) of \( M_i \) as a union of the zero sets \( S_f \), on \( \epsilon < 0 \), [see (7) and (8)].

Step 5: Determine each \( S_i \) and \( M_i \), and then \( M \), by Step 3.

In the following we illustrate the procedure of finding \( M \) by two numerical examples.

C. Numerical examples

The following example is derived from a possible formulation of an absolute stability test of a certain nonlinear system by Jury–Lee criterion (Jury and Lee, 1964).

Example 1. Let \( G = \{ s = (s_1, s_2) \in \mathbb{R}^2 : s_2 > 0 \} \), let \( Q = Q_1 \times Q_2 \), where

\[ Q_1 = \{ q_1 \in \mathbb{C} : |q_1| = 1 \} \]
\[ Q_2 = \{ q_2 \in \mathbb{C} : \text{Re } q_2 \geq 0 \} \cup \{ \infty \}, \]

and let \( P = [0, \infty) \). Let \( f : Q \times P \times G \to \mathbb{R}^2 \cup \{ \infty \} \) be defined by

\[ f(q_1, q_2, p, s_1, s_2) = p[(0.3 - 0.05s_1) + (0.1 - 0.03s_1) \text{Re } q_1] - (0.2 + 0.1 \text{Re } q_1) + 1/s_2 + q_2. \]

We want to find the following set:

\[ M = \{ s \in G : \forall p \in P, \exists (q_1, q_2) \in Q \text{ such that } f(q_1, q_2, p, s_1, s_2) = 0 \}. \]

Determination of the set \( M \). Let \( P_\alpha = [0, \alpha] \). It is easy to see that

\[ P = \bigcup_{\alpha > 0} P_\alpha. \]

Hence

\[ M = \bigcap_{\alpha > 0} M^\alpha, \tag{9} \]

where

\[ M^\alpha = \{ s \in G : \forall p \in P_\alpha, \exists (q_1, q_2) \in Q \text{ such that } f(q_1, q_2, p, s_1, s_2) = 0 \}. \]

The set \( M^\alpha \) is recognized as a generalized zero set of the complex-valued function \( f \). In order to find \( M^\alpha \), and therefore \( M \), we apply the procedure presented above.

Step 1: We set \( \tilde{P} = \mathbb{R} \) and find the set

\[ \tilde{V} = \{(s_1, s_2, p) \in G \times \mathbb{R} : \exists (q_1, q_2) \in Q \text{ such that } f(q_1, q_2, p, s_1, s_2) = 0 \}. \]

In Fruchter et al. (1987a, Example 2) we found this set and we obtained the following result:

\[ \tilde{V} = \{(s_1, s_2, p) \in G \times \mathbb{R} : -0.3 + p(0.4 - 0.08s_1) + 1/s_2 \leq 0 \text{ or } -0.1 + p(0.2 - 0.02s_1) + 1/s_2 \leq 0 \}. \]

Step 2: The set \( \tilde{V} \) can be written as a union of two sets \( \tilde{V}_1 \) and \( \tilde{V}_2 \), namely

\[ \tilde{V} = \tilde{V}_1 \cup \tilde{V}_2, \]

where

\[ \tilde{V}_1 = \{(s_1, s_2, p) \in G \times \mathbb{R} : \varphi_1(p, s_1, s_2) = -0.3 + p(0.4 - 0.08s_1) + 1/s_2 \leq 0 \} \]

and

\[ \tilde{V}_2 = \{(s_1, s_2, p) \in G \times \mathbb{R} : \varphi_2(p, s_1, s_2) = -0.1 + p(0.2 - 0.02s_1) + 1/s_2 \leq 0 \}. \]

Step 3: The set \( M^\alpha \), can be written as a union of two sets \( M^\alpha_1 \) and \( M^\alpha_2 \), namely,

\[ M^\alpha = M^\alpha_1 \cup M^\alpha_2, \]

where

\[ M^\alpha_1 = \{(s_1, s_2) \in G : \forall p \in P_\alpha, \varphi_1(p, s_1, s_2) \leq 0 \} \]

and

\[ M^\alpha_2 = \{(s_1, s_2) \in G : \forall p \in P_\alpha, \varphi_2(p, s_1, s_2) \leq 0 \}. \]

Now, let \( M_1 = \bigcap_{\alpha > 0} M^\alpha_1 \) and \( M_2 = \bigcap_{\alpha > 0} M^\alpha_2 \), then from (9) follows that

\[ M = M_1 \cup M_2. \]

Step 4: The functions \( h_i \), \( i = 1, 2 \) will have the form

\[ h_1(n, p, s) = -0.3 + p(0.4 - 0.08s_1) + 1/s_2 + n \]

and

\[ h_2(n, p, s) = -0.1 + p(0.2 - 0.02s_1) - 1/s_2 + n. \]

Now, the complement of \( M^\alpha_i \) in \( G \), denoted by \( S_i^* \), \( i = 1, 2 \), will be

\[ S_i^* = G \setminus M^\alpha_i = \bigcup_{\epsilon < 0} S_i^\epsilon, \quad i = 1, 2 \]
where
\[ S_{i}^{\alpha, \varepsilon} = \{(s_{1}, s_{2}) \in G: \exists p \in [0, \alpha], \exists \varepsilon \in (-\infty, 0) \text{ such that } h_{i}(p, s_{1}, s_{2}) = 0 \} \]

The complement of \( M_{i} \) in \( G \), denoted by \( S_{i} \), is then given by
\[ S_{i} = G \setminus M_{i} = \bigcup_{\alpha > 0} \bigcup_{\varepsilon < 0} S_{i}^{\alpha, \varepsilon} \quad i = 1, 2 \]

The sets \( S_{1}, i = 1, 2 \), are obtained from \( S_{i}^{\alpha, \varepsilon} \) by taking \( \varepsilon \to -0 \) and \( \alpha \to +\infty \).

Step 5: Determination of \( S_{i}^{\alpha, \varepsilon} \) and \( S_{i} \), \( i = 1, 2 \).

The sets \( S_{i}^{\alpha, \varepsilon} \), \( i = 1, 2 \), are zero sets of the continuously differentiable real-valued functions \( h_{i} \), respectively, relative to \([-\infty, \varepsilon] \times [0, \alpha] \) and \( G \). For finding \( S_{i}^{\alpha, \varepsilon} \) we use the procedure briefed in Section 2. As was mentioned above \( S_{i} \) is obtained from \( S_{i}^{\alpha, \varepsilon} \) by taking \( \varepsilon \to -0 \) and \( \alpha \to +\infty \).

As outlined in this procedure, first one finds the set \( L \). In the present example \( L = L_{0} \cup L_{1} \cup L_{2} \). For finding \( L_{i}, j = 0, 1, 2 \), we apply Theorem 1, with \( d = 1 \) (see Section 2).

Since
\[ \frac{\partial h_{i}}{\partial \varepsilon} = 1 \neq 0, \quad i = 1, 2 \]

we obtain immediately by Theorem 1 that
\[ L_{0} = \emptyset \]

Also, it is readily verified that
\[ h_{i}(-\infty, p, s_{1}, s_{2}) \neq 0, \quad i = 1, 2 \]

Therefore, from (10) and (11) we obtain that in the derivation of \( L_{1} \), we have to consider only the following cases.

In the derivation of \( L_{1} \) we have only the case:
\[ h_{i}(\varepsilon, p, s_{1}, s_{2}) = 0, \quad 0 < p < \alpha, \quad i = 1, 2 \]

and for \( L_{2} \) we have the cases:
\[ h_{i}(\varepsilon, 0, s_{1}, s_{2}) = 0, \quad i = 1, 2 \]
\[ h_{i}(\varepsilon, \alpha, s_{1}, s_{2}) = 0, \quad i = 1, 2 \]

First, we treat the case \( i = 1 \).

The equations of Theorem 1 which correspond to the case (12) are
\[ h_{1}(\varepsilon, p, s_{1}, s_{2}) = -0.3 + p(0.4 - 0.08s_{1}) + 1/s_{2} + \varepsilon = 0 \]

\[ \frac{\partial h_{1}}{\partial p} = 0.4 - 0.08s_{1} = 0 \]

From (15) and (16), taking \( \varepsilon \to 0^{-} \), we obtain
\[ L_{1} = \{(5, 10/3)\} \]

Now, the equations of Theorem 1 for the case (13) are reduced to
\[ h_{1}(\varepsilon, 0, s_{1}, s_{2}) = -0.3 + 1/s_{2} + \varepsilon = 0 \]

Taking \( \varepsilon \to 0^{-} \), we obtain
\[ s_{2} = 10/3 \quad (17) \]

The equations of Theorem 1 for the case (14) are reduced to
\[ h_{1}(\varepsilon, \alpha, s_{1}, s_{2}) = -0.3 + \alpha(0.4 - 0.08s_{1}) + 1/s_{2} + \varepsilon = 0 \]

Taking \( \varepsilon \to 0^{-} \), we obtain
\[ s_{2} = \frac{10}{3} \quad (18) \]

Therefore, from (17) and (18) we obtain
\[ L_{2} = \{(s_{1}, s_{2}) \in G: s_{1} = 5 \text{ or } s_{2} = 10/3\} \]

And in conclusion,
\[ L = \{(s_{1}, s_{2}) \in G: s_{1} = 5 \text{ or } s_{2} = 10/3\} \]

The set \( L \) which is a one-dimensional set, is depicted in Fig. 1 and divides \( G \) into four domains \( D_{i}, i = 1, \ldots, 4 \). In order to decide which of the domains \( D_{i} \) belongs to \( S_{1} \), we choose arbitrary point in each of the domains \( D_{i} \) and check whether these points belong to \( S_{1} \). It is readily verified that
\[ S_{1} = \bigcup_{i=1}^{3} D_{i} \]

The complement of \( S_{1} \) in \( G \) is dashed in Fig. 1 and is given by
\[ M_{1} = \overline{D}_{4} = \{(s_{1}, s_{2}) \in G: s_{1} \geq 5 \text{ and } s_{2} \geq 10/3\} \]

For the case \( i = 2 \) we have similar computations. We obtained that \( S_{1} \subset S_{2} \), therefore \( M_{1} \supset M_{2} \). Hence
\[ \mathcal{M} = M_{1} \cup M_{2} = M_{1} = \overline{D}_{4} \]

And in conclusion,
\[ \mathcal{M} = \{(s_{1}, s_{2}) \in G: s_{1} \geq 5 \text{ and } s_{2} \geq 10/3\} \]

namely the dashed region in Fig. 1.

![Fig. 1. The generalized zero set \( \mathcal{M} = \overline{D}_{4} \) for Example 1 and the connected components of \( G \setminus \mathcal{L} \).](image-url)
Remark 1. If $P$ and $M$ are subsets of $\mathbb{R}$ then $\hat{V}$ is a subset of $\mathbb{R}^2$ [see (2) Section 3B] and $M$ can be found directly from the geometrical interpretation of $\hat{V}$. This is an immediate consequence of Proposition 1(3). In such cases we are able to find $M$ immediately after Step 1, as it will be illustrated in the following example. In consequence, in this case, we do not need the closedness of the set $P$.

Example 2. Let $G = \{s \in \mathbb{R}^*: s > 0\}$, let $Q = Q_1 \times Q_2$, where $Q_1 = Q_2 = [0, \infty)$ and let $P = \mathbb{R}$. Let $f: Q \times P \times G \rightarrow \mathbb{R} \cup \{\infty\}$ be defined by

$$f(q_1, q_2, p, s) = s^{-1} - 6/(36q_1^2 + (5 - q_1)^2) + p(5 - q_1)/(36q_1^2 + (5 - q_1)^2) + q_2.$$

We want to find the following set:

$$\mathcal{M} = \{s \in G: \forall p \in P, \exists (q_1, q_2) \in Q \text{ such that } f(q_1, q_2, p, s) = 0\}.$$

**Determination of the set $\mathcal{M}$**. Let $Q_1^* = [0, \alpha]$ and $Q_2^* = [0, \beta]$. Then, $Q_1 = \bigcup_{\alpha > 0} Q_1^*$ and $Q_2 = \bigcup_{\beta > 0} Q_2^*$. Hence

$$\mathcal{M} = \bigcup_{\alpha > 0} \mathcal{M}^{\alpha, \beta},$$

where

$$\mathcal{M}^{\alpha, \beta} = \{s \in G: \forall p \in P, \exists (q_1, q_2) \in Q_1^* \times Q_2^* \text{ such that } f(q_1, q_2, p, s) = 0\}.$$

According to Remark 1, the set $\mathcal{M}^{\alpha, \beta}$ is recognized as a generalized zero set of the real-valued function $f$. In order to find $\mathcal{M}^{\alpha, \beta}$, and therefore $\mathcal{M}$, we need to use only the Step 1 of the procedure presented above.

**Step 1**: We have $\hat{P} = P = \mathbb{R}$. Let

$$\mathcal{V}^{\alpha, \beta} = \{(s, p) \in G \times \mathbb{R}: \exists (q_1, q_2) \in Q_1^* \times Q_2^* \text{ such that } f(q_1, q_2, p, s) = 0\}.$$

Then

$$\hat{V} = \bigcup_{\alpha > 0} \mathcal{V}^{\alpha, \beta}.$$ 

The set $\mathcal{V}^{\alpha, \beta}$ is recognized as a zero set of the continuously differentiable real-valued function $f$ relative to $Q_1^* \times Q_2^* = [0, \alpha] \times [0, \beta]$ and $G \times \mathbb{R}$. Therefore, for finding $\mathcal{V}^{\alpha, \beta}$, we use the algorithm briefed in Section 2. From $\mathcal{V}^{\alpha, \beta}$, by taking $\alpha \rightarrow +\infty$ and $\beta \rightarrow +\infty$, we obtain $\hat{V}$.

**Determination of $\mathcal{V}^{\alpha, \beta}$ and $\hat{V}$**. As outlined in the procedure, first one finds the set $L$. In the present example $L = L_0 \cup L_1 \cup L_2$. For finding $L_j$, $j = 0, 1, 2$, we apply Theorem 1, with $d = 1$.

(See Section 2.)

Since

$$\frac{\partial f}{\partial a_2} = 1 \neq 0 \quad (18)$$

we obtain immediately by Theorem 1 that

$$L_0 = \emptyset.$$

Therefore, in the derivation of $L$, we have to consider the following cases:

In the derivation of $L_1$, we have the cases

$$f(q_1, 0, p, s) = 0, \quad 0 < q_1 < \alpha \quad (19a)$$

$$f(q_1, \beta, p, s) = 0, \quad 0 < q_1 < \alpha \quad (19b)$$

and for $L_2$ we have the cases

$$f(0, 0, p, s) = 0 \quad (20a)$$

$$f(0, \beta, p, s) = 0 \quad (20b)$$

$$f(\alpha, 0, p, s) = 0 \quad (20c)$$

$$f(\alpha, \beta, p, s) = 0. \quad (20d)$$

It is readily verified that when $\alpha \rightarrow +\infty$, $\beta \rightarrow +\infty$ and $s > 0$, only (19a) and (20a) are meaningful.

Now, in the case (19a) the equation

$$f(q_1, 0, p, s) = s^{-1} - 6/(36q_1^2 + (5 - q_1)^2) + p(5 - q_1)/(36q_1^2 + (5 - q_1)^2) = 0$$

is equivalent to the equation

$$f^*(q_1, p, s) = s^{-1}(36q_1^2 + (5 - q_1)^2) + 6 + p(5 - q_1)/(36q_1^2 + (5 - q_1)^2) = 0. \quad (21)$$

Hence, by Theorem 1 that, in case (19a), we have in addition to (21) the equation

$$\frac{\partial f^*}{\partial q_1} = s^{-1}(36q_1^2 + 5q_1) - p - 0. \quad (22)$$

From (21) and (22) we obtain the solution

$$p^2s^2 - 72ps + 576 + 24s = 0.$$

In the case (20a) we obtain

$$f(0, 0, p = a) = s^{-1} - 6/25 + p/5 = 0$$

or

$$p = 6/5 - 5s^{-1}. \quad (24)$$

From (23) and (24) we obtain

$$L_j = \{(s, p) \in G \times \mathbb{R}: p^2s^2 - 72ps + 576 + 24s = 0 \text{ or } p = 6/5 - 5s^{-1}\}. \quad (25)$$

The set $L_j$ which is a one-dimensional set, is depicted in Fig. 2 and divides $G \times \mathbb{R}$ into four connected domains $D_i$, $i = 1, \ldots, 4$. In order to decide which of the domains $D_i$ belongs to $\hat{V}$, we choose arbitrary points in each of the domains $D_i$ and check whether these points belong to $\hat{V}$. It is readily verified that

$$\hat{V} = \bigcup_{i=1}^4 D_i \cup L.$$

The set $\hat{V}$ is dashed in Fig. 2. From this sketch and Proposition 1 we can conclude immediately
Applying the method of "generalized zero sets location", we find all the feasible set of sectors of nonlinearities, for which absolute robust stability with respect to the uncertainty conditions is ensured, according to the Popov criterion (Popov, 1961).

A. System description

A nominal description of a Lurie continuous-time system with one nonlinear unit is defined by

\[ \dot{x} = Ax + b\phi(a), \quad x = x(t) \in \mathbb{R}^n, \]
\[ A \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^{n \times 1}, \quad a \in \mathbb{R}, \]
\[ \sigma = cx, \quad c \in \mathbb{R}^{1 \times n}. \]  

The nonlinearity \( \phi \) (Lurie nonlinearity) is a continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) satisfying the sector condition:

\[ \phi(0) = 0 \quad \text{and} \quad 0 < \phi(\sigma)\sigma^{-1} < k \quad \text{for} \quad \sigma \neq 0, \]

for a given positive number \( k \), called sector number. Note that this condition restricts the graph of the nonlinearity \( \phi \) to within a specified sector \( S(k) \) which lies in the first and third quadrants of the plane and is bounded by the \( \sigma \)-axis and the line \( y = k\sigma \).

Assume that some or all the entries in \( A, b, c \) are subject to perturbations and are only known to be within given intervals.

A perturbed model of the Lurie continuous system with one nonlinear unit is given by

\[ \dot{x} = A(r)x + b(r)\phi(a) \]  

\[ \sigma = c(r)x, \]

where \( r = (r_1, \ldots, r_q) \), the vector of all uncertainties of the system, may take values in \( \prod_{i=1}^{q} K_i \), where \( r_i \in K_i \) and \( K_i = [\alpha_i, \beta_i], \) \( i = 1, \ldots, q \), are given intervals in \( \mathbb{R} \).

B. Problem statement

Our purpose is to find the complete set of sector numbers \( k \) for which the system (26) is absolutely robustly stable for every \( r \in \prod_{i=1}^{q} K_i \), and for every \( \phi(\sigma) \) which satisfies the sector condition, according to Popov criterion (Popov, 1961). The significance of the solution to this problem, relative to previous works, was explained in the introduction.

C. Problem solution via generalized zero sets location

We will assume that \( A(r) \) of the system (26) is a robust stable matrix. If this is not the case we uncertainties, is studied in this section. Applying the method of "generalized zero sets location", we find all the feasible set of sectors of nonlinearities, for which absolute robust stability with respect to the uncertainty conditions is ensured, according to the Popov criterion (Popov, 1961).
Generalized location and absolute robust stabilization

Corollary 1 Spopov has the form

\[ S_{\text{Popov}} = \{ k \in \mathbb{R}^+: \exists q \in \mathbb{R}, \text{such that} \伽(\omega, r, q, k) > 0, \forall \omega \geq 0 \text{ and } \forall r \in \prod_{i=1}^{q} K_i \} \]

To adopt the above formulation to the generalized set location method, we use an auxiliary variable \( p, p \in [0, \infty) \), in the following way: Let

\[ f(\omega, r, p, q, k) = \伽(\omega, r, q, k) + p. \]

Then

\[ \伽(\omega, r, q, k) > 0 \]

is equivalent to

\[ f(\omega, r, p, q, k) = 0, \forall p \in [0, \infty] \]

and the set \( S_{\text{Popov}} \) becomes

\[ S_{\text{Popov}} = \{ k \in \mathbb{R}^+: \exists q \in \mathbb{R} \text{ such that } f(\omega, r, p, q, k) = 0, \forall \omega \geq 0, \forall r \in \prod_{i=1}^{q} K_i \text{ and } \forall p \in [0, \infty) \} \] (27)

Remark 2. Let \( \omega \geq 0, r \in \prod_{i=1}^{q} K_i, p \in [0, \infty], q \in \mathbb{R} \text{ and } k \in \mathbb{R}^+ \). For stable matrices \( A(r) \), we obtain continuously differentiable functions \( \伽(\omega, r, q, k) \), and therefore, \( f(\omega, r, p, q, k) \).

Remark 3. For each point \( k_0 \) in \( S_{\text{Popov}} \), our system is robust stable with respect to any \( r \) in \( \prod_{i=1}^{q} K_i \) and any nonlinearity \( \psi \) in the sector \( S(k_0) \).

Remark 4. The complement of the set \( \mathcal{M} \) in Example 2 is exactly the set \( S_{\text{Popov}} \) for an unperturbed system.

From (27), we obtain that the complement of \( S_{\text{Popov}} \) in \( \mathbb{R}^+ \) will be

\[ \mathcal{M}_{\text{Popov}} = \mathbb{R}^+ \backslash S_{\text{Popov}} = \{ k \in \mathbb{R}^+: \exists q \in \mathbb{R}, \exists \omega \geq 0, \exists r \in \prod_{i=1}^{q} K_i, \exists p \in [0, \infty] \text{ such that } f(\omega, r, p, q, k) = 0 \} \]

It is easy to see that

\[ \mathcal{M}_{\text{Popov}} = \bigcup_{a > 0} M^a, \]
where
\[ M^\alpha = \{ k \in \mathbb{R}^+ : \forall q \in \mathbb{R}, \exists \omega \in [0, \alpha], \exists \epsilon < 1, \exists r \in \mathbb{R}, \exists t \in [0, \epsilon] \mid (k, q, \omega, r) \in M^\alpha \}. \]

The set \( M^\alpha \) is recognized as a generalized zero set of the continuously differentiable real-valued function \( f \) relative to \( Q = [0, \alpha] \times \left( \prod_{i=1}^{q} K_i \right) \times [0, \infty] \). Therefore, for finding \( M^\alpha \) we may apply the procedure presented in Section 3. Since \( P = \mathbb{R} \) and \( M^\alpha \) is a subset of \( \mathbb{R} \), according to Remark 2 we need only the Step 1 of this procedure. Let
\[ \tilde{V}^\alpha = \left\{ (k, q) \in \mathbb{R}^+ \times \mathbb{R} : \exists \omega, \exists \epsilon < 1, \exists r \in \mathbb{R}, \exists t \in [0, \epsilon] \mid (k, q, \omega, r) \in \tilde{V}^\alpha \right\}, \]

be the zero set of \( f \) relative to \( [0, \alpha] \times \left( \prod_{i=1}^{q} K_i \right) \times [0, \infty] \) and \( \mathbb{R}^+ \times \mathbb{R} \). Then by Proposition 1 we obtain
\[ M^\alpha = \{ k \in \mathbb{R}^+ : \forall q \in \mathbb{R}, (k, q) \in \tilde{V}^\alpha \}, \]

and hence
\[ M_{Popov} = \{ k \in \mathbb{R}^+ : \forall q \in \mathbb{R}, (k, q) \in \tilde{V} \}, \quad (28) \]

where
\[ \tilde{V} = \bigcup_{\alpha > 0} \tilde{V}^\alpha. \]

The set \( \tilde{V} \) can be obtained from \( \tilde{V}^\alpha \) by taking \( \alpha \to +\infty \).

In conclusion, the problem of absolute robust stabilization of a Lurie type continuous-time system with one nonlinearity unit, according to the Popov criterion, is reduced to the problem of locating a generalized zero set in the open subset \( \mathbb{R}^+ \) of \( \mathbb{R} \). This set can be determined immediately after Step 1 of the algorithm presented in the previous section. In this way, the complete set of sector numbers \( k \) can always (for any numbers of uncertainty parameters in the linear part, and all forms of their appearance) be selected from a two-dimensional set:
\[ \mathcal{P} = \mathbb{R}^+ \times \mathbb{R} \setminus \mathcal{V} = \left\{ (k, q) \in \mathbb{R}^+ \times \mathbb{R} : k^{-1} + \text{Re} \left( (1 + j\omega q)G(j\omega, r) \right) > 0, \forall \omega \geq 0, \forall r \in \left( \prod_{i=1}^{q} K_i \right) \right\}. \]

In the following we illustrate the procedure of finding \( M_{Popov} \) by a numerical example.

### D. Numerical example

Consider the continuous system with uncertainty, described in (26), with the transfer function \( G_2(j\omega, r) \) given by
\[ G_2(j\omega, r) = \frac{1}{\lambda(1 + \lambda)(1 + r\lambda + \lambda^2/16)}. \quad (29) \]

The uncertainty parameter \( r \) assumes values in the interval \([0.001, 1/2]\). The nominal value of \( r \) is 0.4/16.

Let \( S_{Popov} \) be the set of points \( k \in \mathbb{R}^+ \) for which this system is absolutely robust stable with respect to any variation of the nonlinearity \( \phi \) in the sector \( S(k) \) and of \( r \) in the interval \([0.001, 1/2]\). By the above discussion we obtain that
\[ S_{Popov} = \{ k \in \mathbb{R}^+ : \exists q \in \mathbb{R} \mid (\omega, r, p, q, k) = 0, \forall \omega \geq 0, \forall r \in [0.001, 1/2] \}, \]

where
\[ f(\omega, r, p, q, k) = k^{-1} + \text{Re} G_2(j\omega, r) + p, \]

and \( G_2(j\omega, r) \) is given by (29), namely
\[ G_2(j\omega, r) = \frac{1}{(1 - \omega^2/16 + r)}(1 - \omega^2/16)^2 + r^2 \omega^2 - 2i \omega(1 + \omega^2)^2((1 - \omega^2/16)^2 + r^2 \omega^2). \]

Evidently, for \( r \) in \([0.001, 1/2]\), \( G_2(\lambda, r) \) has no poles in \( \text{Re} \lambda \geq 0 \), and in conclusion \( f(\omega, r, p, q, k) \) is continuously differentiable (Remark 2).

The complement of \( S_{Popov} \) in \( \mathbb{R}^+ \), denoted by \( \hat{M}_{Popov} \), becomes \( \hat{M}_{Popov} = \mathbb{R}^+ \setminus S_{Popov} = \{ k \in \mathbb{R}^+ : \forall q \in \mathbb{R}, \exists \omega \geq 0, \exists r \in [0.001, 1/2], \exists p \in [0, \infty] \mid (\omega, r, p, q, k) = 0 \}. \)

Setting
\[ \hat{M}_{Popov} = \{ k \in \mathbb{R}^+ : \forall q \in \mathbb{R}, \exists \omega \in [0, \alpha], \exists r \in [0.001, 1/2], \exists p \in [0, \infty] \mid (\omega, r, p, q, k) = 0 \}, \]

such that \( f(\omega, r, p, q, k) = 0 \),
we obtain
\[ \mathcal{M}_{\text{Popov}} = \bigcup_{\alpha > 0} \mathcal{M}_{\text{Popov}}. \]

The set \( \mathcal{M}_{\text{Popov}} \) is the generalized zero set of the continuously differentiable function \( f \), relative to \( Q = [0, \alpha] \times [0.001, 1/2] \times [0, \infty) \), \( P = \mathbb{R} \) and \( G = \mathbb{R}^+ \). Following the discussion in Section 3C, we define
\[ \mathcal{V}^\alpha = \{(k, q) \in \mathbb{R}^+ \times \mathbb{R}: \exists (\omega, r, p) \in [0, \alpha] \times [0.001, 1/2] \times [0, \infty) \text{ such that } f(\omega, r, p, q, k) = 0\}. \]

For finding \( \mathcal{V}^\alpha \) we apply results obtained in (Fruchter et al., 1991a) and briefly summarized in Section 2.

As outlined in the procedure, first one finds the set \( L \). In the present example \( L = L_0 \cup L_1 \cup L_2 \cup L_3 \). For finding \( L_j \), \( j = 0, 1, 2, 3 \), we apply Theorem 1, with \( d = 1 \).

Since
\[ \frac{\partial f}{\partial \omega} = 1 \neq 0 \quad (30) \]
we obtain immediately by Theorem 1 that
\[ L_0 = \emptyset. \]

Also, it is readily verified that
\[ f(\omega, r, +\infty, q, k) \neq 0. \quad (31) \]

Therefore, from (30) and (31) we obtain that in the derivation of \( L \), we have to consider only the following cases:

In the derivation of \( L_1 \) we have the case
\[ f(\omega, r, 0, q, k) = 0, \quad \omega \in (0, \alpha), \quad r \in (0.001, 1/2). \quad (32) \]

In the derivation of \( L_2 \) we have the cases
\[ f(\omega, 0.001, 0, q, k) = 0, \quad \omega \in (0, \alpha) \quad (33) \]
\[ f(\omega, 1/2, 0, q, k) = 0, \quad \omega \in (0, \alpha) \quad (34) \]
\[ f(0, r, 0, q, k) = 0, \quad r \in (0.001, 1/2) \quad (35) \]
\[ f(\alpha, r, 0, q, k) = 0, \quad r \in (0.001, 1/2). \quad (36) \]

Finally, in the derivation of \( L_3 \) we have the cases
\[ f(0, 0.001, 0, q, k) = 0 \quad (37) \]
\[ f(0, 1/2, 0, q, k) = 0 \quad (38) \]
\[ f(\alpha, 0.001, 0, q, k) = 0 \quad (39) \]
\[ f(\alpha, 1/2, 0, q, k) = 0. \quad (40) \]

It is readily verified that when \( \alpha \to +\infty \), and \( k > 0 \), only (32)-(35) and (37)-(38) are meaningful.

Let's consider the case (32). It is easy to see that the equation
\[ f(\omega, r, 0, q, k) = k^{-1} \frac{(1 - \omega^2/16 + r)}{(1 + \omega^2)((1 - \omega^2/16)^2 + \omega^2)} \]
\[ + q \frac{(1 - \omega^2/16 - \omega^2)}{(1 + \omega^2)((1 - \omega^2/16)^2 + \omega^2)} = 0, \quad (41) \]
is equivalent to the equation
\[ f^*(\omega, r, q, k) = k^{-1}(1 + \omega^2)((1 - \omega^2/16)^2 + \omega^2) - (1 - \omega^2/16 + r) - q(1 - \omega^2/16 - \omega^2) = 0. \quad (41) \]

Hence, we obtain by Theorem 1 that, in case (32), we have in addition to (41) the equations
\[ \frac{\partial f^*}{\partial \omega} = k^{-1}(1 - \omega^2/16)^2 + \omega^2 - 1/8((1 - \omega^2/16) + \omega^2) \]
\[ + q\omega - q(\omega/16 + r) = 0 \quad (31) \]
\[ \frac{\partial f^*}{\partial r} = 2\omega^2 - 3/2 \quad (48) \]
\[ \frac{\partial f^*}{\partial q} = 2\omega^2 - 3/2 + q = 0. \quad (47) \]

Finally, for the cases (37) and (38) we obtain
\[ f^*(0, 0.001, q, k) = k^{-1} - 3/2 + q = 0. \quad (48) \]

In conclusion, the set \( L \) is given by the union of (32)-(35) and (37)-(38) denoted by (44) in Figs 3a, b.

Applying Theorem 1 to the cases (33) and (34) we obtain the equations (41) and (42) with \( r = 0.001 \) and \( r = 1/2 \), respectively. These equations yield sets denoted, in Fig. 3, by (45) and (46), respectively.

Applying Theorem 1 to the case (35) we obtain the equations (41) and (43) with \( \omega = 0 \). These equation yield an empty set.

Finally, in the derivation of \( L_3 \) we have the cases
\[ f(0, 0.001, 0, q, k) = 0 \quad (37) \]
\[ f(0, 1/2, 0, q, k) = 0 \quad (38) \]
\[ f(\alpha, 0.001, 0, q, k) = 0 \quad (39) \]
\[ f(\alpha, 1/2, 0, q, k) = 0. \quad (40) \]

It is readily verified that when \( \alpha \to +\infty \), and \( k > 0 \), only (32)-(35) and (37)-(38) are meaningful.

Let's consider the case (32). It is easy to see...
system is absolutely robust stable, according to the Popov criterion, for any nonlinearity $\phi$ in the sector $S(k)$, and any $r \in [0.001, 1/2]$. Following Fig. 3, we obtain that, the maximal sector in which our system is absolutely robust stable is $S(0.6)$.

5. CONCLUSIONS
The generalized zero set introduced in Walach and Zeheb (1982) is extended here to continuously differentiable scalar and vector-valued functions, which depends on several real variables and complex parameters. A new method for locating this set is established here.

A design problem for a continuous nonlinear Lurie type system with a linear part under uncertainty conditions is considered. The complete feasible set of sectors of nonlinearities, for which robust absolute stability is ensured, according to the Popov criterion, is found. The generalized zero set method, proposed here, is applied easily in the solution of this problem.

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